Scales, Sets, and Interval Cycles: A Taxonomy

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Introduction. Recent studies in the theory of scales by Eytan Agmon (1989), Gerald Balzano (1980), Norman Carey with David Clampitt (1989), John Clough with Jack Douthett (1991), Clough with Gerald Myerson (1985, 1986), and Carlton Gamer (1967), which may appear diverse, have in common the central role of a generator—an interval whose repetition ties together all the pitch-classes (pcs) in a set or segment. The set or segment thus formed is commonly known as an interval cycle, either complete in the sense that additional repetitions of the interval yield redundant pcs, or incomplete in the sense that they do not. Drawing largely on properties defined in these studies, we propose a taxonomy for the special categories of pitch-class sets (pcsets) that correspond, collectively, to the interval cycles (complete and incomplete) or to certain conjunctions thereof. The taxonomy is, we believe, interesting in itself, as it addresses complex interactions among features that theorists have defined for different purposes. We believe as well that it strengthens interrelationships among the three concepts in this essay's title: ideas from the theory of scales are cast across pitch-class space, and as a result we discover anew the place of the interval cycle as a pervasive and unifying theoretical notion.

In addition to the common thread of the interval cycle, the cited studies also share a common motivating factor: the desire to explore spaces defined by features of the usual diatonic scale. If we regard a particular property of the diatonic scale as interesting, it is natural to search for other scales with that property, as a means of gaining insight into it. The special status of the generator (or interval cycle) as a feature of many classes of scales will become clear as our study unfolds.

We presume familiarity with 12-pc space, and we hope that our classification will be seen as meaningful for the music of that familiar space. However, our results are intended to apply quite generally to microtonal systems and to all contexts where ordinal distances operate as, for example, in cycles of generic intervals—3ds, 4ths, etc.—*within* scales.

This raises the question of scale versus set. How shall we distinguish between the two? Fortunately, the question is not critical for our purposes here, so it need detain us only briefly. Indeed the question arises only because the literature that we draw upon purports to deal with both scales and sets. It will suffice to say that wherever the concept of a generic interval (2d, 3d, etc.) applies, there lurks a scale-like structure, if not a living scale. It is clear that generic intervals carry musical meaning for the scales of tonal music; it is not clear that they do so for Schoenberg's hexachords. These issues deserve study, but we finesse them in the present paper, making no formal distinction between the terms *scale* and *set* (both refer to unordered sets of pcs) and choosing one term or the other as seems suitable to the context of the discussion.

Our program is as follows: We first define eight features pertaining to scales, all but one of which has been previously defined in the literature. After observing and verifying logical relationships among the features, we identify thirteen sets of features which serve to partition pcsets that have at least one of the features. These thirteen feature-sets (F-sets) support the proposed taxonomy, which amounts to a mapping of pcsets onto the F-sets. The range of individual features over the F-sets then comes into view, followed by proofs of non-existence for pcsets corresponding to sets of features not captured by the thirteen F-sets, and algorithmic approaches to the exhaustive generation of pcsets corresponding to each F-set capable of supporting examples. We close with remarks on the complement relation and additional comments on two particular categories of sets.

1. FEATURES

Example 1 lists the scale features recognized in the present study, with brief definitions and one or two examples of pcsets that exhibit each feature. (Formal notations in Example 1 are explained below.) In some cases we have substituted definitions equivalent to those in the cited sources, to suit our objectives. Pcsets are enclosed in curly brackets; a following subscript indicates the size of the modular chromatic universe if other than 12. To simplify the exposition, and with no loss of precision, we dispense for the most part with the concept of set-class and deal only with literal sets: it is clear that if a pcset has any of the defined features, then all members of the corresponding set-class (under the usual canon of transposition and inversion) have the same features.

There are two additional matters of preamble to the definitions. First, it is necessary to distinguish between rational and irrational generators. Since we deal only with pc space, there is always an underlying modular universe. For convenience, we will speak as though the interval of modularity is the usual 2:1 octave, but the actual size of this interval plays no role in the definitions; nor does it enter into the taxonomy. What *is* significant here is not the size of the modular interval, but whether the generating interval is a rational part of that modular interval. With respect to the modular 2:1 octave, well-known examples of rational and irrational generators are, respectively, the equal tempered 5th and the pure 3:2 5th.[1] As noted below, some of the features apply only to cases where the generator is rational; others apply to both rational and irrational cases. To simplify the exposition, we construct examples only for rational cases, and we imagine these cases to be lodged in equal-tempered systems. The extent to which particular non-equal-tempered tunings are appropriately modeled, in some of their aspects, by equal temperaments, involves a host of interesting questions that we exclude from consideration here.

This leads directly to the second matter of preamble: the distinction between embedded and non-embedded scales. If, as a convenience, we imagine sets with rational generators to be lodged within equal-tempered tunings (as, for example, the usual diatonic in 12-pc space), we are not, as a consequence, obliged to recognize the larger scale steps (whole steps) as divided by (chromatic) pcs that are actually "there" in any relevant musical sense; that is quite a separate issue. The reverse applies to cases with irrational generators. Our conceptions of the Pythagorean heptatonic scale need not be limited to that of a "free-standing" scale. As Carey and Clampitt (1989) have explained, we are free to imagine it embedded within a 12-note Pythagorean system, and that within a still larger Pythagorean system, and so forth. So, like the matter of scale versus set, the matter of embedded versus non-embedded sets or scales lies outside the distinctions of our taxonomy.

With these preliminary notions in place, let us take a brief tour of the definitions of Example 1. *Generated* sets (G-sets) are quite simply those sets associated with interval cycles. We use Marc Wooldridge's (1992) notation for such sets, where

[1]The frequency ratio of the equal-tempered 5th is *irrational;* however it is a *rational* part of an octave: 7/12. On the other hand, the frequency ratio of the pure 5th is *rational,* while it is an *irrational* part of an octave since $\log_2 3/2$ is irrational.

Example 1. Features of scales/sets

Features defined for rational or irrational generators

Feature	Defining characteristic	Example(s)
G generated	Generated by a single interval.	G(12, 7, 5) = {0, 5, 10, 3, 8, 1, 6} = {0, 1, 3, 5, 6, 8, 10} (the usual diatonic) G(7, 3, 4) = $\{0, 4, 1\}_7$ = $\{0, 1, 4\}_7$ (a stack of 5ths in 7-space)
WF well-formed	G-set where each generating interval spans a constant number of scale steps.	G(12, 5, 7) = {0, 7, 2, 9, 4} = {0, 2, 4, 7, 9} (the usual pentatonic)
MP Myhill property	Each generic interval (2d, 3d, etc.) comes in two specific sizes.	G(8, 5, 3) = $\{0, 1, 3, 4, 6\}_8$ <1> = {1, 2}, <2> = {3, 4}
DE distributionally even	Each generic interval comes in either one or two specific sizes.	{0, 1, 6, 7} <1> = {1, 5}, <2> = {6}

Features defined for rational generators only

Feature	Defining characteristic	Example(s)
ME maximally even	Each generic interval comes in either one integer size or two consecutive integer sizes.	{0, 1, 3, 4, 6, 7, 9, 10} <1> = {1, 2}, <2> = {3}, <3> = {4, 5}, <4> = {6}. (the usual octatonic) $\{0, 2, 4\}_7$ <1> = {2, 3} (the triad in 7-space)
DP deep	Every interval class has unique multiplicity.	G(12, 6, 5) = {0, 2, 4, 5, 7, 9} [143250] G (11, 5, 4) = $\{0, 1, 4, 5, 8\}_{11}$ [20341]
DT diatonic	ME-set with $c = 2(d - 1)$ and $c \equiv 0$, mod 4.	G(12, 7, 5) = {0, 1, 3, 5, 6, 8, 10} (the usual diatonic) G(16, 9, 9) = $\{0, 2, 4, 6, 7, 9, 11, 13, 15\}_{16}$ (the next larger diatonic)
BZ Balzano	ME-set with $c = k(k + 1)$ and $d = g = 2k + 1, k \geq 3$.	{0, 1, 3, 5, 6, 8, 10} (k = 3; the usual diatonic) $\{0, 3, 5, 7, 9, 12, 14, 16, 18\}_{20}$ (k = 4)

G(c, d, g) denotes a set of d pcs generated by interval g in a chromatic universe of c pcs. For convenience, we assume throughout that pc 0 is the "origin." Thus $G(c, d, g) = \{0, g, 2g, ..., (d - 1)g\}$, where products are reduced mod c.

Well-formed (WF) scales/sets were defined by Carey and Clampitt (1989). In the example given, the constant number of pcs (four, inclusively) spanned by each generating interval of 7, as well as by the residual "return-to-origin" interval of 8 is evident in the scale segments $<0, 2, 4, 7>$; $<2, 4, 7, 9>$; $<4, 7, 9, 0>$; $<7, 9, 0, 2>$; $<9, 0, 2, 4>$. WF-sets are, with the exception of total-chromatic sets (equal-tempered scales) precisely those defined earlier, in different terms, by Erv Wilson (1975).

The next three definitions are stated in terms of what Clough and Myerson (1986) call the *spectrum* of a generic interval—the set of specific sizes that correspond to the interval. For example, $<1> = \{1, 2\}$ indicates that step intervals come in sizes 1 and 2. Thus, the spectrums of the first three generic intervals in the usual diatonic (the usual 2ds, 3ds, and 4ths) are $<1> = \{1, 2\}$, $<2> = \{3, 4\}$, $<3> = \{5, 6\}$ (the spectrums of the complementary intervals are implied: $<4> = \{6, 7\}$ etc.). *Myhill-property* (MP) sets were defined by Clough and Myerson (1986). *Distributionally even* (DE) sets are isolated, we believe for the first time, in the present paper. *Maximally even* (ME) sets were defined by Clough and Douthett (1991).

Deep (DP) scales/sets were studied by Gamer (1967), who attributes the term and its definition to Winograd (n.d.).

Diatonic (DT) scales/sets, in the present sense, were first isolated by Agmon (1989). A definition of *hyperdiatonic,* proposed by Clough and Douthett (1991), and equivalent to Agmon's definition of *diatonic,* is used here as it comports more naturally with the other definitions of Example 1. A detailed study of the relationship between the two definitions may be found in Agmon (1996).

Finally, the definition of *Balzano* (BZ) sets (Balzano 1980) rests upon an interval k and the next larger interval $k + 1$, whose sum is equal to the size of the generator and whose product is equal to the size of the chromatic universe.

To be sure, a great many additional features of scales and sets have been defined in the literature. Those selected have arisen in the course of studies based on the remarkable set of features possessed by the usual diatonic. But they are by no means the only features to have arisen in this way. Indeed, a number of others deserve mention. Although we exclude them from our system of classification in order to make the problem tractable, some or all of these additional features might well be included in extensions of the present system, or in classifications designed for other purposes.

We will mention four such features. To begin with, there is inversional symmetry. All of the features defined above imply inversional symmetry; so it is everywhere in the classification—a constant property of the scales we study here which therefore plays no role in distinctions among them. The same applies to a second feature, transpositional combination, as developed by Richard Cohn (1991).

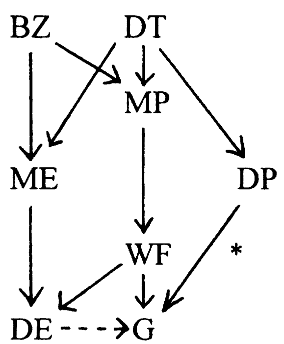
Thirdly, there is prime cardinality. It is easy to see and, we believe, well-known that because the cardinality of the usual diatonic is prime, exhaustive cycles of pcs may be formed by *any* generic interval; hence circles of thirds as well as circles of fifths, and so on. As one of us has discussed elsewhere (Clough 1994), prime cardinality confers also, via the mathematics of primitive roots, the possibility of linking cycles of all generic intervals hierarchically by means of consistent patterns of accent.

Finally, there is that which we shall call Cohn's property, in recognition of his (1996) work describing the class of pcsets associated with what he terms *maximally smooth cycles,* a class studied more formally in Lewin (1996). We return to this property later.

2. IMPLICATIVE RELATIONSHIPS AMONG THE FEATURES

Having defined the several features that we shall be concerned with, we now look at how they combine. As illustrated in Example 2, a number of implicative relationships are present among the features listed in Example 1. For instance, *all* diatonic sets have Myhill property, as symbolized by the arrow from property DT to property MP in Example 2. Further, by following all paths from property DT consistent with the arrows in Example 2, we see that diatonic sets necessarily have *all* the other properties except Balzano's property. (In fact, the diatonic set in 12-pc space is the only diatonic set that *does* have BZ and hence all eight features, as we will show.) If a set has *any* of the listed properties, except ME or DE, it also has G (with the exception of an anomalous class of small-cardinality DP sets). And so forth. Note that the implications shown are all one-way; which is to say that no two features are logically equivalent. The broken arrow from DE to G symbolizes a somewhat different relationship that we will discuss later (see the discussion of Example 5 in Section 3).

Example 2. Implicative relationships among features



*DP implies G if the set is not a form of $\{0, 1, 2, 4\}_6$.

While some of the implicative arrows of Example 2 follow directly from the definitions of the features they connect, others require more involved justification. At this point, we will prove that each of the ten implicative relationships does indeed hold.[2] Readers who wish to skip the following proofs may proceed to Section 3 of the paper. We will proceed with the proofs in the following order:

(1) BZ implies ME.
(2) BZ implies MP.
(3) ME implies DE.
(4) WF implies G.
(5) DT implies ME.
(6) DT implies MP.
(7) MP implies WF.
(8) WF implies DE.
(9) DT implies DP.
(10) DP implies G (if the set is not a form of $\{0, 1, 2, 4\}_6$).

In both (1) and (2), we are dealing with a scale, S, that is BZ. By the definition of BZ, there must be an integer $k \geq 3$ such that S has parameters c and d where $c = k(k + 1)$ and $d = g = 2k + 1$. Recall that two integers x and y are relatively prime if and only if they share no factors greater than 1; i.e., the greatest common divisor of x and y, gcd(x,y), is equal to one, or more compactly, $(x,y) = 1$. Let us begin with the following fact.

Lemma 1.1. *If k is a positive integer, then the integers $k(k + 1)$ and $2k + 1$ are relatively prime.*

PROOF: Suppose that $k(k + 1)$ and $2k + 1$ are not relatively prime. Then there exists a prime p such that $p \mid k(k + 1)$ (i.e., p divides $k(k + 1)$) and $p \mid 2k + 1$. Now, $p \mid k(k + 1) \Rightarrow p \mid k$ or $p \mid k+1$. If $p \mid k$ then $2k+1 \equiv 1 \bmod p$, contradicting $p \mid 2k + 1$. Thus, $p \mid k + 1$. But then $2k + 1 = 2(k + 1) - 1 \equiv -1 \bmod p$, also a contradiction. ■

[2] Although we believe that Example 2 is a complete statement of the pairwise implications among the eight features, we will not give counterexamples to disprove the other possible relationships. Constructing such counterexamples can be quite instructive nevertheless, as readers can verify for themselves.

Lemma 1.1 tells us that $(c,g) = 1$ for all Balzano scales S. Assuming, without loss of generality, that S is at the appropriate transpositional level, we can write S in generated order: $S = \{0, g, 2g, \ldots, (d - 1)g\}$ (expressions evaluated mod c). Therefore, $S = \{0, 2k + 1, \ldots, 2k(2k + 1)\}$. Note that the final generated pc in S is

$$\begin{aligned} 2k(2k+1) &= 4k^2 + 2k \\ &= 2(k^2 + k) + 2k^2 \\ &= 2c + 2k^2 \\ &\equiv 2k^2 \quad \text{mod } c. \end{aligned}$$

Therefore, if we were to generate an "extra" pc in S, we would get

$$\begin{aligned} 2k(2k + 1) + (2k + 1) &\equiv 2k^2 + 2k + 1 &&\text{mod } c \\ &\equiv 2c + 1 &&\text{mod } c \\ &\equiv 1 &&\text{mod } c. \end{aligned}$$

Thus, by Carey and Clampitt's (1989) closure condition, S must be WF; in fact, S is non-degenerate WF since $(c,d) = 1$. (A *degenerate WF scale* is one in which the generating interval and the return-to-origin interval are equal. All other WF scales are *non-degenerate.*) By Carey and Clampitt (1989, 201–2), S must also have MP, proving (2).

In addition, every WF scale has the property that any two adjacent pcs in scale order are spanned by a fixed number of generating intervals, y, where the interval measure from the last pc in generated order to the first is also included in this count. In our present context, g has chromatic length (clen) $2k+1$ and the "left over" interval has clen

$$\begin{aligned} 0 - 2k(2k+1) &\equiv (2k+1) - 1 \quad \text{mod } c \\ &\equiv 2k \text{ mod } c = g - 1. \end{aligned}$$

Therefore, the interval spectrum of a step in S, $\langle 1 \rangle$, is contained in $\{yg, yg - 1\}$, but since S has MP, $\langle 1 \rangle = \{yg, yg - 1\}$. By Clough and Myerson's (1985) Lemma 1, S must have the consecutivity property, and therefore S is ME, proving (1).

Implication (3) is clear using the interval spectrum characterization of ME scales, and (4) is implicit in the definition of WF scales. In addition, (5) follows immediately from the definition of DT, and it will help us prove (6). If S is DT, it must be ME and $(c,d) = 1$. Therefore, by Lemma 1.3 in Clough and Douthett (1991), no spectrum $\langle I \rangle$ of S can contain just one non-zero integer. Thus, $|\langle I \rangle| = 2$ for all I and S has MP, proving (6).

To show (7) and (8), we will use the fact that MP and non-degenerate WF are equivalent properties of scales, stated in Carey and Clampitt (1989) and proved in Carey and Clampitt (n.d.). This equivalence has as a consequence the weaker relation between MP and WF stated in (7). Furthermore, since clearly MP $\Rightarrow$ DE, every non-degenerate WF scale is also DE. We have yet to show that an arbitrary degenerate WF scale, S, is also DE. Note that for such an S, $(c,d) = d$ and $g = c/d$ is a generator. In fact, $S = \{a, a + c/d, \ldots, a + (d - 1)c/d\}$ (all evaluated mod c) for some $a \in 0, 1, \ldots, c - 1$. Therefore

$$\begin{aligned} \langle 1 \rangle &= \{c/d\} \\ \langle 2 \rangle &= \{2c/d\} \\ &\vdots \\ \langle I \rangle &= \{Ic/d\} \text{ for all } I \in 1, \ldots, d-1. \end{aligned}$$

So $|\langle I \rangle| = 1$ and S is DE, completing the proof of (8).

In order to show (9), DT implies DP, we shall prove a more general lemma. Recall that if x is a real number, $\lfloor x \rfloor$ represents the greatest integer less than or equal to x. An expression like "x (mod c)" in the arguments of this section denotes the particular integer between 0 and $c-1$ equivalent to x modulo c.

Lemma 1.2. *Any generated scale S with $(c,g) = 1$ and $d = \lfloor c/2 \rfloor$ or $d = \lfloor c/2 \rfloor + 1$ is a deep scale.*

PROOF: Let S be a generated scale with c, g, and d as above, and let $\mathbf{v} = [v_1, \ldots, v_n]$ be the ic vector for S, with $n = \lfloor c/2 \rfloor$. Note that any scale embedded in a universe of size $c = 3$ or less has only one entry in its ic vector, and thus is trivially deep. Thus, we need consider only those scales with $c > 3$.

If $g > c/2$, then $-g$ mod c is also a generator of S, so we can assume without loss of generality that $0 < g < c/2$. It is clear that $v_g = d-1$, since S is some transposition of $S' = \{0, g, 2g,$

..., $(d-1)g\}$ (all mod c) and $dg \not\equiv 0 \bmod c$ for $c > 3$. Knowing that this "generalized circle of fifths" ordering of S has precisely one break in it leads us to the broader observation that

$$\begin{aligned} v_{\theta(mg)} &\geq d - 1 - (m-1), \text{ for } m = 1, 2, ..., d-1 \\ &\geq d - m \end{aligned}$$

$$\text{where} \quad \theta_c(x) = \begin{cases} x \bmod c, & \text{if } 0 \leq x (\bmod c) \leq c/2 \\ c - x \bmod c, & \text{if } c/2 < x(\bmod c) < c. \end{cases}$$

Now, if we could show that this inequality is actually an equality, then we would be well on the way to showing that all of the entries of **v** are distinct. Let us first show that the above inequality applies to $d - 1$ elements of **v**.

Claim 1.2.1. *$\theta_c(mg)$, where $m = 1, ..., d - 1$, generates $d - 1$ distinct integers.*

PROOF—CLAIM: Certainly, mg (mod c) yields $d - 1$ distinct integers for $m = 1, ..., d - 1$ because $d - 1 < c$ and $(c,g) = 1$. Suppose $\theta_c(m_1g) = \theta_c(m_2g)$ for some $m_1, m_2 \in 1, ..., d - 1$. Then either: (1) $m_1g \equiv m_2g \bmod c \Rightarrow m_1 = m_2$, and we are done; or (2) $m_1g \equiv -m_2g \bmod c \Rightarrow m_1 \equiv - m_2 \bmod c$. However, $d = \lfloor c/2 \rfloor$ or $d = \lfloor c/2 \rfloor + 1$, so $d - 1 \leq c/2$. In order for $m_1 \equiv - m_2 \bmod c$, we must have $m_1 = m_2 = d - 1 = c/2$, and our claim is proved.

The claim shows us that the multi-set $W = \{v_{\theta(mg)} \mid m \in 1, ..., d - 1\}$ contains $d - 1$ entries from the ic vector **v** of S, although we are not yet sure that W consists of $d - 1$ *distinct* integers. Let us look at two cases based on the cardinality of S.

<u>Case 1:</u> $d = \lfloor c/2 \rfloor + 1$

Here, $\lfloor c/2 \rfloor = d - 1$, so W contains all of the entries of **v**. The sum of the entries must equal

$$\binom{d}{2} = \frac{d(d-1)}{2} = 1 + 2 + ... + d - 1.$$

But in order to get this sum, the inequality above must be an equality: $v_{\theta(mg)} = d - m$ for $m = 1, ..., d - 1$, since all entries of **v** must be less than d. If any of the entries of **v** were larger than the minimum required by the inequality, the sum of the entries would be too high. Thus, the entries of **v** are the integers from 1 to $d - 1$, and S is deep.

<u>Case 2:</u> $d = \lfloor c/2 \rfloor$

Now, W contains all but one entry of **v**. Again the sum of the entries is $1 + ... + d - 1$, and again, this requires that $v_{\theta(mg)} = d-m$ for $m = 1, ..., d - 1$. The remaining entry of **v** must be zero since the sum of the other entries (which are all non-zero) is equal to the total sum of the entries of **v**. Thus, the entries of **v** are the integers from 0 to $d - 1$, and S is deep. ■

By definition, $d = c/2 + 1$ and $(c,d) = 1$ for all diatonic scales S, which by Clough and Douthett's (1991) Theorem 3.1 implies that S is generated. Furthermore, since $d > c/2$, the generator of S must be relatively prime to c. Therefore, the conditions of Lemma 1.2 hold for diatonic scales, and invoking this Lemma, we have shown (9).

Implication (10), DP implies G if the scale is not a form of $\{0, 1, 2, 4\}_6$, is somewhat more difficult to show and will require proving a number of lemmas first. Fortunately, these lemmas are also useful as further formal characterizations of deep scales mentioned in passing by Gamer (1967). The first two lemmas below work towards determining the possible diatonic cardinalities and the makeup of ic vectors of deep scales, and the next two use this information to analyze the structure of such scales.

Lemma 1.3. *If a scale S has parameters c and d, where $c \geq 5$, and d is an entry in the ic vector of S, then there are at least two equal entries in that ic vector.*

PROOF: Let $\mathbf{v} = [v_1, v_2, ..., v_n]$ be the ic vector of S, and note that $n = \lfloor c/2 \rfloor$. Suppose that there exists an $i \in 1, 2, ..., n$, such that $v_i = d$. Note that if c is even, $i \neq c/2$, since the maximum value of the "tritone" entry of **v** is d/2. For all $x \in S$, $x + i \pmod c \in S$. Therefore, for all $x \in S$, $x + 2i \pmod c \in S$ as well. Let us consider two cases.

<u>Case 1:</u> $2i \neq c/2$

Since $x + 2i \pmod c \in S$ for all $x \in S$, $v_j = d$, where

$$j = \begin{cases} 2i, \text{ if } 2i \leq \lfloor c/2 \rfloor \\ c - 2i, \text{ if } 2i > \lfloor c/2 \rfloor. \end{cases}$$

Thus, we are guaranteed two distinct "d" entries in **v** if $i \neq j$. Suppose $i = j$. Since $i \neq 2i$, we have $i = c - 2i \Leftrightarrow c = 3i$, and $2i > \lfloor c/2 \rfloor$.

Let $\mathcal{O} = \{y \in S \mid y+1 \pmod c \in S\}$. Since $c > 2$, $v_1 = |\mathcal{O}|$. Let $\mathcal{O}' = \{y' \in S \mid y' + 1 + i \pmod c \in S\}$. In this case, because $c = 3i \Rightarrow i = c/3 < c/2 - 1$ for $c \geq 5$, we have $v_{i+1} = |\mathcal{O}'|$ as long as $i + 1 \neq c/2 \Leftrightarrow i \neq 2$. But if $i = 2$, then S is some transposition of $\{0, 2, 4\}_6$ or $S = \{0, 1, 2, 3, 4, 5\}_6$, yielding ic vectors of [0 3 0] and [6 6 3], respectively, and we are done.

So assuming that $i \neq 2$, take some $y \in \mathcal{O}$. Then $y + 1 \in S \Rightarrow y + 1 + i \in S$ (all mod c) since $v_i = d$. Therefore, $y \in \mathcal{O}'$ and $\mathcal{O} \subseteq \mathcal{O}'$. Similarly, if $y' \in \mathcal{O}'$, then $y' + 1 + i \pmod c \in S \Rightarrow y' + 1 + i + 2i \equiv y' + 1 \pmod c \in S$, and $y' \in \mathcal{O} \Rightarrow \mathcal{O}' \subseteq \mathcal{O}$. Thus, $\mathcal{O} = \mathcal{O}'$ and $v_1 = v_{i+1}$.

<u>Case 2:</u> $2i = c/2$

Here, $4i = c$. Let $\mathcal{T} = \{x \in S \mid x + 1 \pmod c \in S\}$. Note that $v_1 = |\mathcal{T}|$ since $c > 2$. Let $\mathcal{T}' = \{x' \in S \mid x' + 1 + i \pmod c \in S\}$. Because $c = 4i \Rightarrow i = c/4 \leq c/2 - 1$ for $c \geq 5$, $v_{i+1} = |\mathcal{T}'|$ as long as $i+1 \neq c/2 \Leftrightarrow i \neq 1$. But $i = 1 \Rightarrow c = 4$, so $i \neq 1$.

If $x \in \mathcal{T}$, then $x + 1 \in S \Rightarrow x + 1 + i \in S$ (all mod c) and $x \in \mathcal{T}'$ which implies $\mathcal{T} \subseteq \mathcal{T}'$. Similarly, if $x' \in \mathcal{T}'$, then $x' + 1 + i \pmod c \in S \Rightarrow x' + 1 + i + 3i \equiv x' + 1 \pmod c \in S$, and $x' \in \mathcal{T}$. Thus $\mathcal{T}' \subseteq \mathcal{T} \Rightarrow \mathcal{T}' = \mathcal{T}$ and $v_1 = v_{i+1}$. ■

Lemma 1.4. *Suppose S is a deep scale in a universe of cardinality $c \geq 5$. Then $d = \lfloor c/2 \rfloor$ or $d = \lfloor c/2 \rfloor + 1$, and entries of the ic vector of S are $d - 1, d - 2, \ldots, 1$, and possibly 0 (if and only if $d = \lfloor c/2 \rfloor$).*

PROOF: Suppose the ic vector of S is $\mathbf{v} = [v_1, \ldots, v_n]$ where $n = \lfloor c/2 \rfloor$. Since S is deep, all of the entries in **v** are unique. Let

$$\sigma = \sum_{i=1}^{n} v_i \, .$$

Note that

$$\sigma = \binom{d}{2} = \frac{d(d-1)}{2} = 1 + 2 + \ldots + d - 1.$$

We know that the sum $(d - 1) + (d - 2) + \ldots + (d - n)$ is an upper bound on σ from the unique ic multiplicity property of S and the fact that $v_i < d$, for $i = 1, 2, \ldots, n$ (by Lemma 1.3). Therefore,

$$(d - 1) + \ldots + 1 \leq (d - 1) + \ldots + (d - n).$$

Thus, n must be d or $d - 1 \Rightarrow d = \lfloor c/2 \rfloor$ or $\lfloor c/2 \rfloor + 1$, and the two sums are equal. Hence if $d = \lfloor c/2 \rfloor$ then the entries of **v** are $\{d - 1, d - 2, \ldots, 0\}$ and if $d = \lfloor c/2 \rfloor + 1$ the entries are $\{d - 1, d - 2, \ldots, 1\}$. ■

Lemma 1.5. *Suppose S is a deep scale with $c \geq 5$ and ic vector $v = [v_1, \ldots, v_n]$ where $v_q = d - 1$. Then S is generated by q if and only if $(c,q) = 1$.*

PROOF: Note that the existence of a q such that $v_q = d - 1$ is guaranteed by Lemma 1.4.

($\Leftarrow$) Suppose $(c,q) = 1$. There is a unique element of S, a_1, such that $a_1 - q \pmod c \notin S$, and another unique element, $a_2 \in S$, such that $a_2 + q \pmod c \notin S$. Let $S' = \{a_1 + iq \pmod c \mid i = 0, 1, \ldots, k$, where $a_1 + kq \equiv a_2 \bmod c\}$. Since $(c,q) = 1$, such a k will exist, and S' will consist of $k + 1$ distinct elements of S $\Rightarrow k \leq d - 1$. If $k < d - 1$, then we can find an element $x \in S - S'$. Since $x \neq a_1$ or a_2, we have $\{x + nq \pmod c \mid n = 0, 1, \ldots\} \subseteq S \Rightarrow S = \{0, 1, \ldots, c-1\}$ which is not possible. Thus $k = d - 1 \Rightarrow S' = S \Rightarrow S$ is generated by q.

($\Rightarrow$) Suppose S is generated by q and $(c,q) > 1$. Then $d \leq c/2$. By Lemma 1.4, we are left with two possibilities for the cardinality of S:

(1) $d = c/2$, if c is even

(2) $d = (c - 1)/2$, if c is odd.

If $d = c/2$, then $(c,q) = 2$, and S consists of all non-negative even integers less than c or all non-negative odd integers less than c. Therefore, $v_2 = d$, and by Lemma 1.3, S is not deep, contradicting our original premise.

If c is odd, then $(c,q) \geq 3 \Rightarrow d \leq c/3$. By (2), $(c - 1)/2 \leq c/3 \Rightarrow c \leq 3$, contradicting our restriction on c. ■

Lemma 1.6. *Suppose S is a deep scale with $c \geq 5$ but $c \neq 6$, and ic vector $\mathbf{v} = [v_1, ..., v_n]$ where $v_q = d - 1$. Then $(c,q) = 1$.*

PROOF: Recall that $n = \lfloor c/2 \rfloor$ and that if c is even, $v_n \leq d/2$. In fact, from Lemma 1.4, if c is even and subject to the restrictions stated above, then $d/2 < d - 1$. Thus, $q < c/2$.

Suppose $(c,q) > 1$, and consider the set $A = \{mq \pmod c \mid m = 0, 1, ...\}$. Then $|A| \leq c/2$. Now, from Lemma 1.4 we know that $d = \lfloor c/2 \rfloor$ or $\lfloor c/2 \rfloor + 1$. So as long as $d \neq (c - 1)/2$, $|A| \leq d$. But if $d = (c - 1)/2$ then c must be odd, therefore $(c,q) > 2 \Rightarrow |A| \leq c/3 = (2d + 1)/3 \leq d$ for all $d \in \mathbb{Z}^+$. So we have shown that $|A| \leq d$ for all d. With this information, we will construct a quasi-cyclic representation of S.

Since $v_q = d-1$, we can find a $\alpha_1 \in S$ such that there exists a "complete q-cycle"[3] embedded in S: $F_1 = \{\alpha_1, \alpha_1 + q, \alpha_1 + 2q, ..., \alpha_1 + \beta q \equiv \alpha_1\}$ (all elements taken mod c), where $|F_1| = \beta = |A|$. (The existence of at least one such full q-cycle is guaranteed: if there were no such cycle, there would be at least two "partial q-cycles" (since $|A| \leq d$) in S, implying the existence of at least two elements of S which were not T_q related to any other elements of S.) Note that if $F_1 = S$, then $v_q = d$, which is not possible; so $F_1 \neq S$. Find an element $x \in S - F_1$. If possible, find such an x so that $x = \alpha_2$ is a member of another full q-cycle in S, $F_2 = \{\alpha_2, \alpha_2 + q, ..., \alpha_2 + \beta q \equiv \alpha_2\}$ (all mod c). If no such x exists, choose x such that it begins the unique partial q-cycle in S, $P = \{x, x + q, ..., x + bq\}$ (all mod c), where $x + bq \not\equiv x \bmod c$. In this latter case, we have necessarily enumerated all of the elements of S: $S = F_1 \cup P$, where this union is disjoint. Continuing this process iteratively, S can be partitioned into $r > 1$ full q-cycles, $F_1, ..., F_r$, and one partial q-cycle P. Note that P must be non-empty, or else v_q would be equal to d.

Let Y be a pc set. Define UP(Y) to be the set of all unordered pairs of distinct elements in Y. Furthermore, define the function VCTR on unordered pairs of disjoint pc sets in c-space such that $\mathrm{VCTR}(A,B) = [w_1, w_2, ..., w_{\lfloor c/2 \rfloor}]$, where w_i is the number of pairs (a,b), $a \in A$ and $b \in B$, such that $\mathrm{int}(a,b) = i$ or $c - i$.

[3]This is unrelated to David Clampitt's (1997) Q-cycle.

Note that indeed VCTR(A,B) = VCTR(B,A). Finally, define $\mathrm{VCTR}_i(A,B)$ to be the entry w_i in the vector VCTR(A,B).

As we have shown above, $S = F_1 \cup F_2 \cup ... \cup F_r \cup P$, where these component sets are all disjoint. Let $X_S = \{F_1, ..., F_r, P\}$. If $\mathbf{V_Y}$ is defined to be the ic vector for the pc set Y, then

$$\mathbf{v} = \mathbf{v_S} = \mathbf{vF_1} + \mathbf{vF_2} + ... + \mathbf{vF_r} + \mathbf{v_P} + \sum_{(A,B) \in UP(X_S)} \mathbf{VCTR(A,B)} \quad \textbf{(Eq. 1)}$$

Consider $v_1 = |T_1(S) \cap S|$. Because $(c,q) > 1$, for any set $X \in X_S$, $|T_1(X) \cap X| = 0$. Similarly, $|T_{1+q}(X) \cap X| = 0$ for all $X \in X_S$. However, it is not immediately clear that $v_{1+q} = |T_{1+q}(S) \cap S|$ since $1+q$ may be greater than $n = \lfloor c/2 \rfloor$, in which case v_{1+q} is undefined, or it could be equal to c/2, implying that $v_{1+q} = |T_{1+q}(S) \cap S|/2$. Let us therefore determine the conditions under which $v_{1+q} = |T_{1+q}(S) \cap S|$.

Claim 1.6.1. *(1) If c is odd and $c \geq 5$, then $q + 1 \leq \lfloor c/2 \rfloor$.*
(2) If c is even and $c > 6$, then $q + 1 < c/2$.

PROOF—CLAIM: In both cases, we know that $q < c/2$. We will treat each case separately.

(1) Since $q + 1 < c/2 + 1$ and c is odd, we know that $q + 1 \leq (c+1)/2$. Suppose $q + 1 = (c+1)/2$. Then $c = 2q + 1$. But q and $2q + 1$ are relatively prime: if p divides q then it cannot divide $2q + 1$ unless $p = 1$. This contradicts our assumption that $(c,q) > 1$. Therefore $q + 1 \leq (c-1)/2 = \lfloor c/2 \rfloor$.

(2) Since $q + 1 < c/2 + 1$ and c is even, we know that $q + 1 \leq c/2$. Suppose $q + 1 = c/2$ with $c > 6$, and consider $(c,q) = (c, c/2 - 1)$. Find $p > 1$ such that $p|c$ and $p|(c/2 - 1)$. Then there exists an integer k such that $c = kp$ and $p|(kp/2 - 1)$. There exists another integer k′ where $kp/2 - 1 = k'p \Leftrightarrow p(k - 2k') = 2 \Leftrightarrow p = 2$. Thus, $(c,q) = 2$. Therefore, $|F_1| = c/2$. If $d = c/2$, then $S = F_1$, which is a contradiction; so $d = c/2 + 1$. Therefore, P must consist of a single element s, and $S = F_1 \cup \{s\}$. Hence, $v_2 = d - 1$ which implies that $q = 2$ because the entries of $\mathbf{v}$ are unique. By our assumption, we have $c = 6$, a contradiction. This proves our claim.

Thus, under the conditions that $c \geq 5$ and $c \neq 6$, we are assured that $v_{1+q} = |T_{1+q}(S) \cap S|$.

Summarizing the results above, we have

$$v_1 = |T_1(S) \cap S|, \ |T_1(X) \cap X| = 0 \quad \text{for all } X \in X_S, \text{ and}$$
$$v_{1+q} = |T_{1+q}(S) \cap S|, \ |T_{1+q}(X) \cap X| = 0 \quad \text{for all } X \in X_S.$$

Combining this information with Eq. 1, we have

$$v_1 = \sum_{(A,B) \in UP(X_S)} VCTR_1(A,B), \text{ and} \qquad \textbf{(Eq. 2)}$$

$$v_{1+q} = \sum_{(A,B) \in UP(X_S)} VCTR_{1+q}(A,B). \qquad \textbf{(Eq. 3)}$$

We will now show that the sums in Equations 2 and 3 are equal.

Consider $(P,F_i) \in UP(X_S)$ where $i \in 1, \ldots, r$. Then we have

$$VCTR_1(P,F_i) = |T_1(P) \cap F_i| + |T_1(F_i) \cap P|, \text{ and}$$
$$VCTR_{1+q}(P,F_i) = |T_{1+q}(P) \cap F_i| + |T_{1+q}(F_i) \cap P|.$$

Now, in general, $|A \cap B| = |T_k(A) \cap T_k(B)|$ for any k. In addition, in the present context $F_i = T_q(F_i)$ since F_i consists of a full q-cycle. Therefore, $T_1(F_i) = T_{q+1}(F_i)$, and

$$|T_1(P) \cap F_i| = |T_{1+q}(P) \cap T_q(F_i)| = |T_{1+q}(P) \cap F_i|, \text{ and}$$
$$|T_1(F_i) \cap P| = |T_{1+q}(F_i) \cap P|.$$

Thus,

$$VCTR_1(P,F_i) = VCTR_{1+q}(P,F_i). \qquad \textbf{(Eq. 4)}$$

Similarly, consider $(F_i,F_j) \in UP(X_S)$ for $i, j \in 1, \ldots, r$ $(i \neq j)$. Then

$$VCTR_1(F_i,F_j) = |T_1(F_i) \cap F_j| + |T_1(F_j) \cap F_i|, \text{ and}$$
$$VCTR_{1+q}(F_i,F_j) = |T_{1+q}(F_i) \cap F_j| + |T_{1+q}(F_j) \cap F_i|.$$

By arguments analogous to the previous case, we get

$$VCTR_1(F_i,F_j) = VCTR_{1+q}(F_i,F_j). \qquad \textbf{(Eq. 5)}$$

Equations 4 and 5 show us that the terms of the summations in Equations 2 and 3 are identical, proving that

$$\sum_{(A,B) \in UP(X_S)} VCTR_1(A,B) = \sum_{(A,B) \in UP(X_S)} VCTR_{1+q}(A,B).$$

Thus $v_1 = v_{1+q}$, contradicting our original premise that S is deep. Therefore, our assumption that $(c,q) > 1$ was incorrect and $(c,q) = 1$. ■

The majority of the work towards showing implication (10) is now complete. We need only combine the information about deep scales gleaned by the lemmas and clean up the leftover business related to possible anomalies in small universes. This is done in the following theorem.

Theorem 1.7. *If S is deep and S is not a transpositional form of* $\{0, 1, 2, 4\}_6$*, then S is generated.*

PROOF: Let us begin by assuming that S is embedded in a universe of size $c \geq 5$ and $c \neq 6$. Suppose S has ic vector $\mathbf{v} = [v_1, \ldots, v_n]$ where $v_q = d - 1$ (by Lemma 1.4, we know that such a q exists). By Lemma 1.6, $(c,q) = 1$. Therefore by Lemma 1.5, S is generated by q and we are done.

Note that *every* scale of diatonic cardinality 1, 2, $c - 1$, or c is trivially generated. In particular, every scale in a universe of size $c \leq 4$ is necessarily generated. Thus, we are left only with the $c = 6$ case.

If the deep scale S is embedded in a universe of size $c = 6$, then $d = 3$ or 4 by Lemma 1.4. For $d = 3$, there are four non-transpositionally equivalent forms, listed here with their ic vectors:

$\{0,1,2\}_6$ [210]
$\{0,2,4\}_6$ [030]
$\{0,1,3\}_6$ [111]
$\{0,1,4\}_6$ [111]

Of these, only $\{0,1,2\}_6$ is deep, and it is also generated by 1.

For c = 6 and d = 4, there are three non-transpositionally equivalent forms:

$\{0,1,2,3\}_6$ [321]

$\{0,1,2,4\}_6$ [231]

$\{0,1,3,4\}_6$ [222]

Of these, $\{0,1,2,3\}_6$ and $\{0,1,2,4\}_6$ are deep, but only $\{0,1,2,3\}_6$ is generated (again by 1). ■

The appearance of a single exception to the rule that DP ⇒ G may seem curious, but its existence follows fairly naturally from the argument presented in the proof of Lemma 1.6, specifically from Claim 1.6.1. In colloquial terms, scales existing in small universes are sometimes simple enough to have the properties of DP and G without fulfilling more complicated structural requirements normally associated with these properties in the context of a larger universe. Every scale in universes of $c \leq 5$ is generated, so $c = 6$ is a kind of "breaking point" for the property G. The breaking point for DP is slightly higher, resulting in the anomalous transposition class in $c = 6$.

We end this section with a corollary that brings together much of the work on deep scales done here and presents an alternative characterization of sufficiently large deep scales based on the parameters c, d, and g.

Corollary 1.8. *Take any scale S in a universe of size $c \geq 5$ where S is not transpositionally related to $\{0, 1, 2, 4\}_6$. Then S is deep if and only if S is generated by g where $(c,g) = 1$ and S has cardinality $d = \lfloor c/2 \rfloor$ or $\lfloor c/2 \rfloor + 1$.*

PROOF: (⇐) Follows immediately from Lemma 1.2.

(⇒) If S is deep with $c \geq 5$, then by Lemma 1.4, $d = \lfloor c/2 \rfloor$ or $\lfloor c/2 \rfloor + 1$. Furthermore, if $c \geq 5$ but $c \neq 6$, then Lemma 1.6 locates a q such that $(c,q) = 1$ and Lemma 1.5 guarantees that q generates S. Finally, if $c = 6$ but S is not a transpositional form of $\{0, 1, 2, 4\}_6$, then the proof of Theorem 1.7 shows that S must be generated by 1. ■

3. FEATURE SETS

Assuming that Example 2 is exhaustive in its portrayal of implicative relationships among the features, one might suspect that any set of features consistent with these implications—one can count twenty such (non-empty) sets—would be instantiated by a set of pcs somewhere, in *some* universe. But this is not the case. As we will show, there are sets of features consistent with Example 2 that are incapable of instantiation by actual pcsets. We are in fact able to construct pcsets consistent with just thirteen of the twenty sets of features that are consistent with Example 2. These peculiarities of instantiation seem to arise from the fact that the various features address properties that are not all commensurable with one another.

The twenty sets of features consistent with Example 2, which we call *potential feature sets* or *PF-sets,* are summarized in Examples 3a and 3b, the former listing the instantiated PF-sets, or *F-sets,* and the latter listing the uninstantiated PF-sets. In both examples, the sets are listed in the order of greatest to fewest number of features present. Each PF-set is characterized by those features (indicated by ★'s) necessary to imply *all* of the features (indicated by ★'s and ✔'s) held by its class, based on the implicative relations in Example 2. Those thirteen PF-sets which can be instantiated—the F-sets—are assigned numbers.

F-sets. Let us now look at the PF-sets that *are* instantiated by pcsets. From Example 2, we see that of all the listed features, only G and DE might appear in isolation, and indeed they do appear so, as we will soon see. From the example we also see that there are three pairs of features that might appear in the absence of other features: namely (ME, DE), (G, DP), and (G, DE); however only the first two of these are associated with actual pcsets. At the other end of the spectrum—sets with *many* of the cited features—we see from Example 2 that BZ and DT may exist independently of each other; so it is not clear from the example that any pcsets would have all eight features. But, as mentioned above, there is in fact a class of sets—the usual diatonic set-class in 12-pc space—that *is* associated with all

Example 3. The PF-sets

a. F-sets

F-set number	G	DE	WF	ME	MP	DP	BZ	DT	Number of features	Example
1	✔	✔	✔	✔	✔	✔	★	★	8	{0, 2, 4, 5, 7, 9, 11} (usual diatonic)
2	✔	✔	✔	✔	✔	✔		★	7	$\{0, 2, 3, 5, 7\}_8$ (diatonic in 8-space)
3	✔	✔	✔	✔	✔		★		6	$\{0, 2, 5, 7, 9, 11, 14, 16, 18\}_{20}$ (BZ, k=4)
4	✔	✔	✔	★	★	★			6	$\{0, 2, 4\}_7$ (triad in 7-space)
5	✔	✔	✔	★	★				5	{0, 3, 5, 8, 10} (usual pentatonic)
6	✔	✔	★	★		★			5	$\{0, 2\}_4$
7	✔	✔	✔		★	★			5	$\{0, 1, 4, 5, 8\}_{11}$
8	✔	✔	✔		★				4	$\{0, 5, 6, 11, 12\}_{13}$
9	✔	✔	★	★					4	$\{0, 1, 2, 3, 4, 5, 6\}_7$ (total chromatic)
10	✔					★			2	$\{0, 2, 3, 6\}_7$ (DP, 3 step sizes)
11		✔		★					2	{0, 1, 3, 4, 6, 7, 9, 10} (octatonic)
12		★							1	{0, 1, 4, 5, 8, 9} (hexatonic, DE alone)
13	★								1	{0, 3, 5, 10} (G alone)

b. Uninstantiated PF-sets

G	DE	WF	ME	MP	DP	BZ	DT	Number of features
✔	✔	✔	✔	✔	★	★		7
✔	✔		★		★			4
✔	✔	★			★			4
✔	✔	★						3
★	✔		★					3
✔	★				★			3
★	★							2

eight features. Furthermore, the usual diatonic set-class is unique in this respect, as shown in the following theorem.

Theorem 2.1. *G(12,7,7) is the only scale with all eight properties.*

PROOF: Given the implicative relationships among features as shown in Example 2, any scale that is DT and BZ will necessarily possess all the remaining properties.

For DT scales, $d = g = (c/2) + 1$ and $c \equiv 0 \bmod 4$.

For BZ scales, $c = k(k + 1)$ and $d = g = 2k + 1$ for some integer k ($k > 2$).

For scales that are both DT and BZ, then,

$$(c/2) + 1 = 2k + 1$$

or, by substitution,

$$(k(k + 1)/2) + 1 = 2k + 1$$

$$(k^2 + k)/2 = 2k$$

$$k^2 + k = 4k$$

$$k + 1 = 4$$

$$k = 3.$$

G(12,7,7) is therefore the only scale that is both DT and BZ, and thus the only scale with all eight properties. ■

In addition to the five instantiated sets of features mentioned thus far (two singletons, two pairs, and one complete set) there are eight other PF-sets that are represented by actual pcsets, for a total of thirteen F-sets. All of these are listed in Example 3a, along with a sample pcset corresponding to each F-set. We conceive the relationship between the pcsets and the F-sets to be a many-to-one mapping from the former onto the latter: a given pcset corresponds to one and only one F-set—the F-set with precisely all of its features.[4]

Let us scan the roster of F-sets in Example 3a, and note an example of each. As shown above, F-set 1 is the only F-set corresponding to a unique set-class (under transposition and inversion)—the usual diatonic. Also this F-set represents the only intersection between DT and BZ. F-set 2 contains all DT-sets except the usual one; the diatonic set in 8-space is shown in the example. F-set 3, in parallel fashion to F-set 2, contains all BZ-sets except the usual diatonic; the BZ-set in 20-space ($k = 4$) is shown.

F-set 4, first isolated by Agmon (1989), includes the triad in 7-space (shown), the 7th-chord in the same space, and other sets where c is odd, $d \leq \lfloor c/2 + 1 \rfloor$, and $g = 2$. F-set 5 represents the usual pentatonic (given as an example) and, as it turns out, all complements of other DT-sets; these are Clough and Douthett's (1991) *hyperpentatonic* sets.

Deferring comment on F-sets 6–8 for the moment, we note that F-set 9 embraces equal-tempered chromatic scales in all universes (the one in 7-space is shown); these are Carey and Clampitt's (1989) degenerate WF-sets.

F-sets 11 and 12, correspond respectively, to the sharply restricted cases with DE and ME only or DE only, exemplified here by the usual octatonic and the *hexatonic* set (after Cohn 1996). Note that these two F-sets alone lack G; we comment further on F-sets 11 and 12 below, in two different contexts.

Two F-sets correspond to G-sets with only one additional feature (as in F-set 10) or no additional features (F-set 13). It is notable that the first of these corresponds to all and only DP-sets with three step sizes.

It is difficult to characterize the remaining F-sets (numbers 6–8) in any way that speaks to intuition. They simply are what they are. (We are indebted to David Clampitt for pointing out the instantiation of F-set 7.) It is remarkable that F-set 6 is the only F-set aside

[4]As mentioned above, we are restricting our investigation to the special category of pcsets that are composed of interval cycles or certain conjunctions thereof. By "certain conjunctions thereof," we refer precisely to combinations of interval cycles that produce DE-sets. These pcsets correspond to scales that have at least one of the eight features discussed, with the exception noted in Theorem 1.7: the scales transpositionally related to the deep scale $\{0, 1, 2, 4\}_6$. The theorem showed that these scales are not G, and since $<2> = \{2, 3, 4\}$, these scales are not DE either. From Example 2, we conclude that these scales have DP and only DP. While it would have been possible to specify another F-set category for this scale class, we have chosen instead to treat it as an anomaly, preferring to remain consistent with our original goal of categorizing only those scales that are composed of interval cycles.

from F-set 1 corresponding to a finite (indeed very small) collection of sets; we comment further on this collection below.

The data of Example 3a, in the columns labeled "features," are re-formatted in Example 4, which lists the F-sets that include each feature. With these data in hand, we can explore the complex relationships within and among conglomerates of sets defined by means of logical operators over the features. Suppose, for example, that we wish to look at pcsets that have DE and not WF. We see that such pcsets correspond to F-set 11 (ME and DE) and F-set 12 (DE only). Given that the octatonic and the hexatonic are examples, respectively, of these two F-sets, we may reasonably guess that the pcsets consistent with the logical expression DE and not WF are precisely those that are symmetrical under (non-trivial) transposition and symmetrical under inversion (Messiaen's modes, for example), some of which are ME and others not. Below we will show that this is in fact the case.

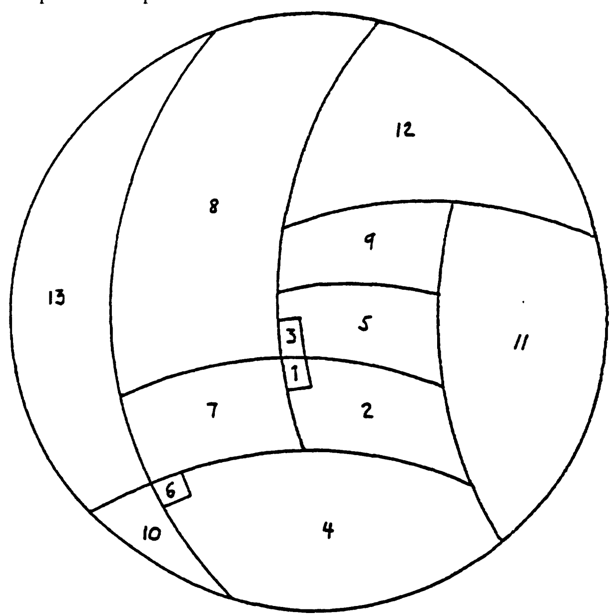
The above conglomerate of pcsets, defined by a simple logical expression over the features, and corresponding to just two F-sets, is relatively easy to intuit. As an aid in grasping larger conjunctions of F-sets that correspond to more complex logical expressions, the representation of Example 5 is useful. The diagram allows us to visualize the ranges of particular features and combinations of features over the F-sets. Each F-set is assigned an area within the circle in such a way that, for any particular feature, the areas corresponding to F-sets with that feature are contiguous. Thus, for example, feature DP, found in F-sets 1, 2, 4, 6, 7, and 10 (see Example 4) may be viewed as a territory corresponding to these F-sets (shaded areas of Example 6) thus providing a sub-taxonomy—a classification of Garner's deep scales. Similarly, a sub-taxonomy of WF scales is evident in Example 7, where we see the extent of such scales in the shaded areas of F-sets 1–9. The intersection of the shaded areas in Examples 6 and 7 (corresponding to F-sets 1, 2, 4, 6, and 7) would contain all the scales with both DP and WF. Fortunately this intersection is a contiguous area on the map, but such is not always the case. The logical expression "G and ME and not DT," for instance, corresponds to F-sets 3, 4, 5, 6, and 9; we see in Example 8 that the conglomerate of pcsets satisfying that expression is non-contiguous on the map.

What of the "ground" of Example 5—sets that lie outside the circle? Given that F-set 13 contains sets with G only, we conclude that all sets outside the circle are non-G sets. The

Example 4. Extents of features across F-sets

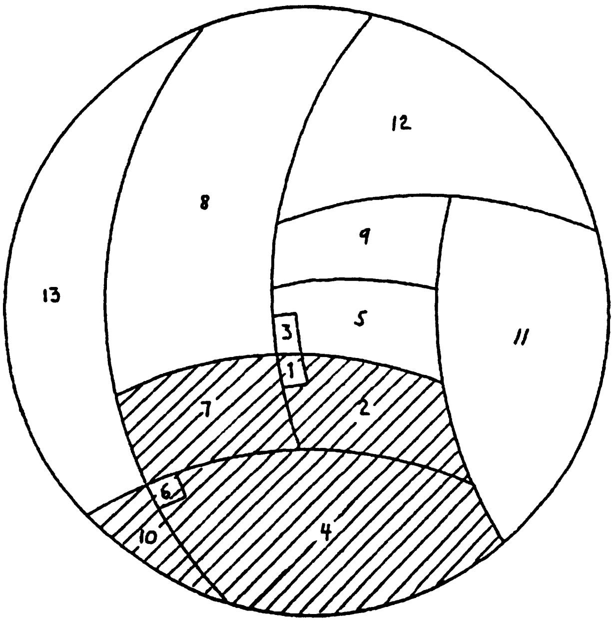
G:	1–10, 13
DE:	1–9, 11–12
WF:	1–9
ME:	1–6, 9, 11
MP:	1–5, 7–8
DP:	1–2, 4, 6–7, 10
BZ:	1, 3
DT:	1–2

Example 5. "Map" of F-sets



Example 6. Territory of DP scales

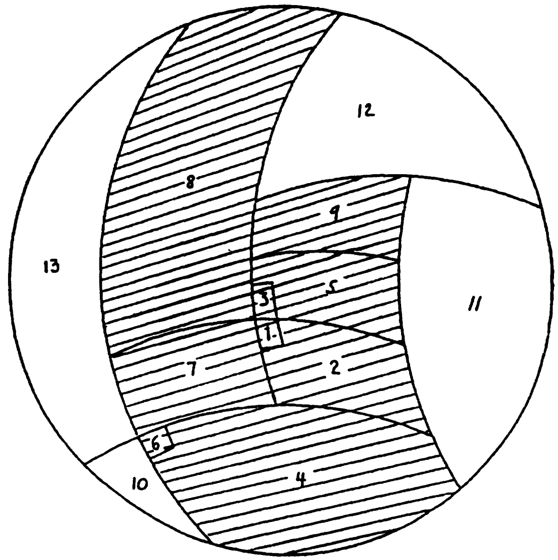
F-set	features, (X) = not X
1	all
2	(BZ)
4	(BZ, DT)
6	(BZ, DT, MP)
7	(BZ, DT, ME)
10	G, DP



converse (that all sets within the circle have G) would appear to be false, as F-sets 11 and 12 lack G. However, pcsets corresponding to these F-sets are, in fact, generated, in the sense that they are composed of smaller, identically generated sets that divide the octave equally, where the smaller sets are themselves disposed at equal intervals. This is the expanded sense of "generated" indicated by the broken arrow in Example 2. The pcsets of F-sets 11 and 12 are thus examples of sets held invariant under non-trivial transposition—a special category of such sets, however. In this sense of "generated," then, Example 5 is a map of precisely all and only generated pcsets—that is to say, pcsets based on interval cycles.

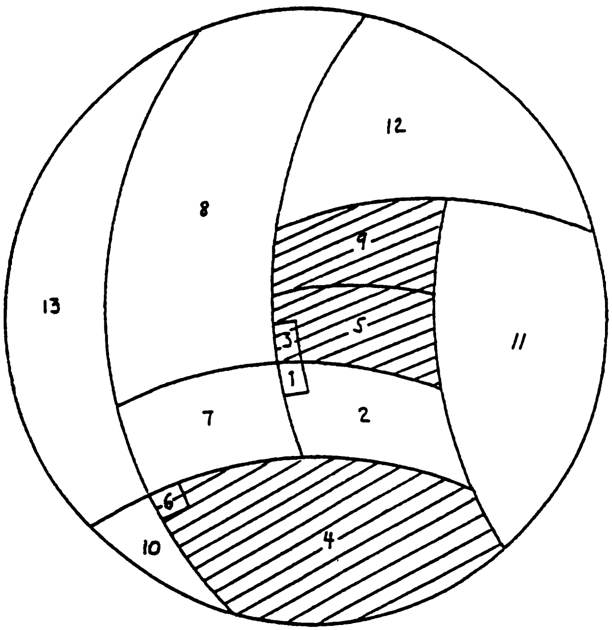
Example 7. Territory of WF scales

F-set	features, (X) = not X
1	all
2	(BZ)
3	(DP, DT)
4	(BZ, DT)
5	(DP, BZ, DT)
6	(MP, BZ, DT)
7	(ME, BZ, DT)
8	(ME, DP, BZ, DT)
9	(MP, DP, BZ, DT)



Example 8. Territory of scales with G and ME and *not* DT

F-set	features, (X) = not X			
3		(DP,		DT)
4			(BZ,	DT)
5		(DP,	BZ,	DT)
6	(MP,		BZ,	DT)
9	(MP,	DP,	BZ,	DT)



Toward further exemplification of the F-sets, and as an interesting exercise for its own sake, let us examine a familiar scenario. We put *g*, a generator, equal to interval seven and set it running in 12-pc space. The results, shown in Example 9, are comparable to those derived from a similar exercise in Carey and Clampitt (1989).

Uninstantiated PF-sets. We shall now investigate the nature of the seven PF-sets that are consistent with the implicative diagram in Example 2, but that seem to lack concrete examples. Our goal will be to prove conclusively that these PF-sets possess internal inconsistencies rendering them uninstantiable in any chromatic universe. Example 10 presents the seven PF-sets grouped into three categories. Each category will be investigated separately, and all details of the analysis pertaining to a given category will apply to all PF-sets within that category. Also included in Example 10 are the general strategies that will be used to approach each of the three PF-set categories. Proving the non-existence of the Category 1 PF-set is simply a matter of showing that a scale cannot be both BZ and DP without also being DT. All of the Category 2 PF-sets are DE and G, but none are WF; so if we can show that being both DE and G forces a scale to also be WF, then we would be sure that no Category 2 PF-sets can be exemplified. The two Category 3 PF-sets are WF, but neither is ME *or* MP. Thus, if we knew that all WF scales were ME and/or MP then these two PF-set possibilities would be ruled out.

Example 9. Tour of U_{12} with g = 7

n	G(12, n, 7)	F-set
0	{ }	13: G ?
1	{0}	9: G, WF, ME, DE
2	{0, 7}	5: G, MP, WF, ME, DE
3	{0, 2, 7}	5
4	{0, 2, 7, 9}	13
5	{0, 2, 4, 7, 9}	5
6	{0, 2, 4, 7, 9, 11}	10: G, DP
7	{0, 2, 4, 6, 7, 9, 11}	1: all features
8	{0, 1, 2, 4, 6, 7, 9, 11}	13
9	{0, 1, 2, 4, 6, 7, 8, 9, 11}	13
10	{0, 1, 2, 3, 4, 6, 7, 8, 9, 11}	13
11	{0, 1, 2, 3, 4, 6, 7, 8, 9, 10, 11}	5
12	{0, 1, 2, 3, 4, 5, 6, 7, 8, 9 10, 11}	9

Before the demonstrations of these facts are given, a general comment about the proofs is in order. The arguments given

Example 10. Four categories of the seven uninstantiated PF-sets

Category	Characterizing PF-set features	Strategy
1	BZ, DP	Show that BZ and DP imply DT
2	ME, DP ME, G DP, DE DE, G	Show that DE and G imply WF
3	WF, DP WF	Show that WF implies ME and/or MP

below use little mathematics beyond modular arithmetic and basic number theory. Portions of some of the proofs make use of elementary group theory. The style of the proofs is meant to be as colloquial as possible without sacrificing any mathematical rigor. By presenting the arguments in this way, it is hoped that the reader who does not follow every detail of the proof can still comprehend the general approach, the underlying methodology, and the intermediate results which comprise the "background structure" fueling the line-by-line momentum within each demonstration. Readers who wish to skip the proofs, however, should proceed to Section 4.

Category 1

Category 1 is the simplest case to deal with because both BZ and DP have characterizations based solely on the values of c, d, and g. The definition above represents BZ in just this way, and the alternate form of the definition of DP given in Corollary 1.8 can be used in this context since every BZ scale requires c to be at least 12, which is greater than 6. We are ready for our first result.

Theorem 3.1. *All scales S that are both BZ and DP are also DT.*

PROOF: Recall that the definition of BZ requires $c = k(k + 1)$ and $d = g = 2k + 1$, where k is some integer greater than 2. Since c is a product of consecutive integers, it must be the product of an even integer with an odd. Any such product is necessarily even, so we know that c is even for all BZ scales.

From the Corollary 1.8 characterization of DP and the fact that c is even, d must be $c/2$ or $c/2 + 1$. Combining this with the definition of BZ, we know that

$$d = k(k + 1)/2 \text{ or } k(k + 1)/2 + 1.$$

Furthermore, $d = 2k + 1$; so substituting for d we get the following:

$$\begin{array}{rclcrcl} 2k+1 &=& k(k+1)/2 & \text{OR} & 2k+1 &=& k(k+1)/2+1 \\ 4k+2 &=& k^2+k & & 4k &=& k^2+k \\ 0 &=& k^2-3k-2 & & 0 &=& k^2-3k \\ & & & & 0 &=& k(k-3) \\ k &=& \dfrac{3 \pm \sqrt{17}}{2} & \text{OR} & k &=& 0 \text{ or } 3 \end{array}$$

The solutions on the left are non-integers, leaving 0 and 3 as the only possibilities for k. But the definition of BZ stipulates that k is greater than 2, so k must be 3. Substituting $k = 3$ back into the original equations yields

$$c = 12 \text{ and } d = g = 7.$$

From our knowledge of scales, we know exactly what these parameters imply: the familiar diatonic in the usual chromatic universe. Since d is odd, $c = 2(d - 1)$, and we know the usual diatonic to be ME, it is also DT. ■

Note that the proof of Theorem 3.1 actually shows that not only does a scale with BZ and DP have to be DT, but it has to be the *usual* diatonic, i.e., the major scale embedded in the 12-note universe. This should not surprise us, given that the usual diatonic is the unique scale with all eight features (Theorem 2.1).

Category 2

To show that Category 2 PF-sets cannot exist, we will proceed by proving that all such scales have properties that force them to be WF. The strategy will be to show that the lack of MP in Category 2 F-sets enables one to conclude that such scales are not just generated, but cyclically generated. In other words, knowing that the Category 2 scale S is generated implies that S is of the form {a, a 1 g, a 1 2g, ..., a1(d 2 1)g} (all evaluated mod c), but we will go even further to prove that the "last" element in generated order is g chromatic steps from the "first" element:

$$[a + (d - 1)g] + g \equiv a \pmod{c}$$

This is obviously not true of all generated scales; one need go no further than our familiar diatonic scale to find a counterexample. Any cyclically generated scale is necessarily WF; in fact, such a scale is degenerate well-formed, by definition.

The proof of the theorem in question is possibly the most complicated of the paper. Unfortunately, tidy parameter relationships which helped us in our proof of Theorem 3.1 are not readily available in the context of Category 2 PF-sets. Arguing by cases, the proof of Theorem 3.2 builds up the machinery necessary to compare the representation of S in scalar order to that in generated order.

Theorem 3.2. *All scales S that are both DE and G are also WF.*

PROOF: Suppose S is both DE and G. Then it has a generator g such that $S = \{a, a + g, a + 2g, ..., a + (d - 1)g\}$ (all mod c) for some integer a, $0 \leq a < d$. For purposes of cleaner notation, we will assume without loss of generality that $a = 0$. Note that if a were not 0, then we could transpose S by $-a$ chromatic steps to obtain the desired form. Since the scalar features that we are investigating here are preserved under transposition (in fact, all eight features have this property), such a transformation will not affect the relevant properties of the scale. Thus, we will take S to be $\{0, g, 2g, ..., (d-1)g\}$ in generated order, each expression evaluated modulo c. In scalar order, $S = \{d_0, d_1, ..., d_{d-1}\}$. Throughout the paper, the subscripts of d will always be taken modulo d unless otherwise stated. Note that since $0 \in S$, $d_0 = 0$.

If S has MP, then it must be WF (and therefore is not a Category 2 PF-set), and we would be done. Thus, for the remainder of the proof, we will assume that S does not have MP.

Since S is DE, we know that every generic interval (i.e., diatonic interval) comes in one or two chromatic sizes. But since S does not have MP, there must be at least one dlen I that only comes in one chromatic size. Say $\langle I \rangle = \{a\}$. Therefore, we have

$$d_{j+I} - d_j \equiv a \pmod{c} \qquad \text{(Property 1)}$$

for all $j \in \mathbb{Z}_d$.

Consider the subset of S,

$$A = \{d_0, d_I, d_{2I}, ...\}.$$

There are two important things to notice about A. First, since all adjacent elements of A (in this representation, not necessarily in scalar order) are separated by I diatonic steps, they are all multiples of a:

$$d_0 \equiv 0, d_I \equiv a, d_{2I} \equiv 2a, ..., d_{jI} \equiv ja \pmod{c}$$

Second, the cardinality of A is dependent on the relationship between the integers I and d. We will rely on the former observation to gain insight into the inner structure of S, and we will utilize the latter to break up our proof into manageable pieces. The three cases to consider are: (1) I and d are relatively prime, (2) I divides d (i.e., d is a multiple of I), and (3) I and d have a common factor greater than 1 and less than I. We investigate the cases in this order.

Case 1: $(I, d) = 1$

If I and d are relatively prime (with $I < d$), then I is a generator of the group $\mathbb{Z}_d$. Therefore, the set of subscripts of the elements d_j in A are all of the integers between 0 and $d - 1$ inclusive. Thus, as unordered sets,

$$A = \{0, a, 2a, ..., (d - 1)a\} \text{ (all mod c)} = S.$$

Since the diatonic distance between $d_{(d-1)I} \equiv (d-1)a \bmod c$ and $d_{dI} = d_0 = 0$ is I, by Property 1 the chromatic distance between the two is a. Thus, S is cyclically generated by $g = a$.

Because the generator a always spans I diatonic steps, S must be WF.

Case 2: (I, d) = I

Since I divides d, we can find a β, $0 < \beta < d$, such that $d = \beta I$ (this is an equality, not just a modular equivalence). Since β is the smallest number of I's that need to be summed so as to arrive at an integer equivalent to 0 mod d, the cardinality of A, $|A|$, must be β. Therefore,

$$A = \{d_0 = 0, d_I, d_{2I}, \ldots, d_{(\beta-1)I}\}.$$

In this particular case, it is important to notice that the subscripts of d do not need to be reduced modulo d: they are all strictly less than d, since $\beta I = d$. Therefore, the representation of A above is in scalar order: $jI < iI$ implies $d_{jI} < d_{iI}$, for all integers i, j where $0 \leq i,j < \beta$. From these observations, we also know that $\beta a \equiv 0$ mod c, because each dlen I corresponds to a clen a.

Given these facts about A, we can construct the following matrix representation of S:

$$S = \begin{bmatrix} d_0 = 0 & d_I & d_{2I} & \cdots & d_{(\beta-1)I} \\ d_1 & d_{I+1} & d_{2I+1} & \cdots & d_{(\beta-1)I+1} \\ d_2 & d_{I+2} & d_{2I+2} & \cdots & d_{(\beta-1)I+2} \\ \vdots & \vdots & \vdots & & \vdots \\ d_{I-1} & d_{2I-1} & d_{3I-1} & \cdots & d_{\beta I-1} \end{bmatrix} \quad \text{(Matrix 1)}$$

The following three observations can be made about Matrix 1:

(1) Matrix 1 is a scalar ordered representation of S with no duplicate elements. The ordering is apparent by reading down the columns and skipping back up to the first element in the next column after reaching the bottom. Since each column contains I elements and since there are β rows, the number of matrix entries is βI, which is d (see above), the cardinality of S.

(2) The first row of the matrix is the scalar ordered representation of A.

(3) Since the dlen between any two horizontally adjacent elements in the matrix is I, the clen between such elements must be a. Therefore, all rows of the matrix are ordered transpositions of A.

Recall that S is of the form $\{d_0 = 0, g, 2g, \ldots, (d-1)g\}$ (all mod c) for some generator g. Since $A \subseteq S$ and A is cyclically generated by a, we can view A as being cyclically generated by $kg \equiv a$ mod c, where k is the number of generating intervals, g, it takes to get from d_{jI} to $d_{(j+1)I}$. Since $\beta a \equiv 0$ mod c, we have

$$\beta(kg) \equiv 0 \text{ mod c.}$$

This equation suggests that Matrix 1 may be able to give us some information about the nature of g. If we could say a little more about βk, the equation could lead us right to our desired result. However, to get the requisite mileage out of the equation, we will need to fiddle a bit with the ordered representation of S.

Choose the smallest integer r, $0 < r < d$, such that $rg \equiv \alpha a$ mod c for some α where $0 < \alpha < \beta$. In other words, r is the smallest integer such that rg (mod c) is a multiple of a (mod c); or equivalently, r is the smallest integer such that rg (mod c) is an integer somewhere in the top row of Matrix 1, i.e., somewhere in A. We know that such an r exists because $A \subseteq S = \{rg \text{ (mod c)} \mid 0 \leq r < d\}$. The reason we are suddenly so concerned with this r is that it turns out that A is composed completely of multiples of rg. This is proven in the following lemma.

Claim 3.2.1. *Using the definition of r just given,* $A = \{nrg \text{ (mod c)} \mid 0 \leq n < \beta\}$.

PROOF—CLAIM: Since $mrg \equiv m\alpha a$ (mod c), we know that all multiples of rg, mrg, are in A, which is composed solely of multiples of a, modulo c. We will proceed to show that A is composed *only* of multiples of rg, using a proof by contradiction.

First, recall that all elements of A are integral multiples of g mod c. Suppose that there exists a positive integer γ such that γg (mod c) $\in$ A, but $\gamma g \not\equiv nrg$ (mod c), for any positive integer n. Now, since A is composed of multiples of a, there exists an integer α' such that $\gamma g \equiv \alpha' a$ (mod c). Because r was the minimal element with this property and $\gamma \neq r$, $\gamma > r$. Using the division algorithm, we can express γ uniquely as

$$\gamma = Qr + R, \text{ where } 0 < R < r.$$

Note that since γ is not a multiple of r, R is non-zero. Substituting into the equation above, we have the following:

$$\begin{aligned}
(Qr + R)g &\equiv \alpha' a && \text{(mod c)}\\
Qrg + Rg &\equiv \alpha' a && \text{(mod c)}\\
Rg &\equiv \alpha' a - Q(rg) && \text{(mod c)}\\
Rg &\equiv \alpha' a - Q\alpha a && \text{(mod c)}\\
Rg &\equiv (\alpha' - Q\alpha)a && \text{(mod c)}
\end{aligned}$$

So, Rg is a multiple of a, with $R < r$. But this contradicts the assumption that r was the smallest integer that yielded a product rg that was a multiple of a. Therefore, no such γ exists and all elements of A are multiples of rg, which proves our claim.

Call $A = \{0, rg, 2rg, ..., (\beta-1)rg\}$ (all mod c) the generated order of A. Armed with this new information about A, we can permute the columns of Matrix 1 based on the generated order of A to obtain a representation (mod c) of S which is more comparable to the generated order of S:

$$S = \begin{bmatrix}
d_0 = 0 & rg & 2rg & \dots & (\beta-1)rg\\
d_I & d_I+rg & d_I+2rg & \dots & d_I+(\beta-1)rg\\
d_2 & d_2+rg & d_2+2rg & \dots & d_2+(\beta-1)rg\\
\vdots & \vdots & \vdots & & \vdots\\
d_{I-1} & d_{I-1}+rg & d_{I-1}+2rg & \dots & d_{I-1}+(\beta-1)rg
\end{bmatrix} \quad \text{(Matrix 2)}$$

Applying arguments to Matrix 2 analogous to those made with respect to Matrix 1, we know that

$$\beta(rg) \equiv 0 \text{ mod c}.$$

Furthermore, Matrix 2 gives us the ability to say something about βr, which we were not able to do with βk above. Between any two consecutive elements in the generated order of A, there are r elements in the generated order of S (if there were fewer, r would not be minimal, as required):

$$\begin{aligned}
S &= \{0, g, 2g, ..., (r-1)g, rg, (r+1)g, ..., 2rg, ..., (\beta-1)rg, ..., (d-1)g\}\\
A &= \{0, \qquad rg, \qquad 2rg, ..., (\beta-1)rg\}
\end{aligned}$$

(all elements mod c)

From this observation, it is clear that the cardinality of S, d, is equal to βr. Thus, from the equation above, we know that $\beta(rg) \equiv (\beta r)g \equiv dg \equiv 0$ mod c. Hence, adding one more g onto the last element of S in generated order, $(d-1)g$, brings us back to the first element, 0. Therefore, S is cyclically generated by g, so it must be a degenerate WF scale.

<u>Case 3:</u> $(I,d) = h < I$

It turns out that every Case 3 scale is also a Case 2 scale. We will show this using a group theoretic argument.

By elementary number theory, we know that $(I,d) = h$ implies that $(I/h, d/h) = 1$. Thus, I/h is a generator of the group $\mathbb{Z}_{d/h}$. Therefore, I generates a subgroup of $\mathbb{Z}_d$ isomorphic to $\mathbb{Z}_{d/h}$, and $|A| = d/h$. This subgroup consists of the integers that are subscripts of d in the set $A = \{d_0, d_I, d_{2I}, ...\}$.

Since h divides d, h also generates a subgroup of $\mathbb{Z}_d$ isomorphic to $\mathbb{Z}_{d/h}$. Because there is only one such subgroup, the subgroups generated by I and h are one and the same. Therefore we can write A as

$$A = \{d_0, d_h, d_{2h}, ..., d_{(d/h-1)h}\}.$$

Since I is a generator of the subgroup, there exists an integer k such that $h \equiv kI$ (mod d). Thus,

$$\langle h \rangle = \langle kI \rangle = \{ka \text{ (mod c)}\}.$$

So, $\langle h \rangle$ is a one-element spectrum where $(h,d) = h$. This reduces Case 3 to Case 2, and we are left with a degenerate WF scale. ■

One sees an immediate consequence of the proof of Theorem 3.2 by observing that once the possibility of MP is counted out, each of the three cases leads to a degenerate WF scale.[5] Thus, we have the following corollary.

Corollary 3.3. *For every scale S that is both G and DE, the following four statements are equivalent:*

(A) S is not MP.

(B) S has at least one one-element spectrum.

(C) S is degenerate WF.

(D) All of the interval spectrums of S consist of one element.

PROOF: We will show that (A) ⇒ (B) ⇒ (C) ⇒ (D) ⇒ (A).

(A) ⇒ (B)

Since S is DE, all generic intervals in S come in one or two sizes. If S is not MP, there must be at least one generic interval, I, that does not come in two sizes. Hence, <I> must consist of one element.

(B) ⇒ (C)

S is G, DE, and has at least one one-element spectrum and therefore is not MP. The proof of Theorem 3.2 shows that S must be cyclically generated, and hence degenerate WF.

(C) ⇒ (D)

S, degenerate WF ⇒ S is cyclically generated. Take the generator to be g. Since the g-multiples mod c constitute a complete cycle through all the elements of S (under the appropriate transposition), we can find a k < d such that $d_1 = kg + d_0$. In a WF scale, the number of generating intervals that span a given diatonic interval is constant.[6] Therefore, $d_{j+1 \pmod d} - d_j \equiv kg$ mod c for all non-negative integers j < d. Hence, <1> = {kg (mod c)}. From this, it is clear that <i> = {ikg (mod c)}, for all positive integers i < d.

(D) ⇒ (A)

Clear from the definition of MP. ■

Category 3

Both of the Category 3 PF-sets are WF, but neither is MP or ME. Hence, it will be sufficient to show that the WF property always draws MP and/or ME along for the ride. One way of approaching this problem is to employ a powerful observation about WF scales made in Carey and Clampitt (1989, 191). The authors show that one can always find a positive integer k, k < d, such that the map f: S → $\mathbb{Z}_d$, where jg → jk (mod d) (j ∈ $\mathbb{Z}_d$), is one-to-one and onto, and $jg \equiv d_{f(jg)} = d_{jk \pmod d}$ (mod c). By relating the generated and scalar orderings of a WF scale via a well-behaved mapping, the authors have set up the possibility of establishing representations of S that are more mathematically pliable than the case-based structures that were needed for the proof of Theorem 3.2. However, it turns out that the application of such a mapping is not necessary to achieve our result, as our labors in the proof of Theorem 3.2 will pay off in the form of a simple argument to dispose of Category 3 PF-sets.

Theorem 3.4. *All scales S that are WF must be either ME, MP, or both.*

PROOF: Suppose S is WF. If S has MP, then we are done, so we will assume that S does not have MP, and we will show that S must have ME.

By the implicative diagram of Example 2, S must be both DE and G. Since S is not MP, we can invoke Corollary 3.3, which tells us immediately that S is degenerate WF and that all the interval spectrums of S are made up of one element. Thus, S must be ME. ■

[5]See Carey and Clampitt (1989), 200–2. Corollary 3.3 is consistent with Carey and Clampitt's assertion that MP is equivalent to non-degenerate WF.

[6]This is true even for non-degenerate WF scales, where the "leftover" interval from g(d − 1) to $d_0 = 0$ is counted as a "generating interval." The proof of this fact is somewhat technical and is omitted here for the sake of brevity.

Note that it was the "over-determination" in the proof of Theorem 3.2 that enabled us to achieve this result. We were only required to show that S was WF in Theorem 3.2, but the proof showed that S was degenerate WF, as long as S was not MP.

4. ENUMERATION

It is one thing to characterize a family of pcsets in terms of a particular F-set and another to show how the family's pcsets may be exhaustively enumerated. Two of the thirteen F-sets have finite and in fact very limited memberships—F-set 1 contains the usual diatonic as its sole constituent, as demonstrated in Theorem 2.1, and F-set 6 turns out to encompass only a finite number of somewhat trivial classes that can be easily enumerated. Each of the remaining eleven F-sets, however, has an infinite number of members (given that we have not placed any general restrictions on the size of the chromatic universe), making the matter of exhaustive enumeration appear far less tractable. We will begin with F-set 6 and then turn our attention to the remaining eleven F-sets.

F-SET 6. Since F-set 6 is DE and G, but does not have MP, by Corollary 3.3 it must be degenerate WF. Therefore, there exists a positive integer k mod d, such that $<1> = \{k\}$. We can assume without loss of generality that

$$S = \{0, k, 2k, ..., (d-1)k\},$$

where the entries do *not* need to be taken modulo c. Note that $c = kd$ (again an equality).

Note also that $<I> = \{Ik\}$ for all $I = 1, ..., d - 1$. Therefore, all chromatic intervals represented in S will occur d times, except for the tritone (if c is even and a tritone is represented in S), which will occur d/2 times in the interval vector. Thus, in order for S to be DP, only one non-tritone interval can be represented in the interval vector. In addition, to avoid duplicate zero entries, only one interval can be absent in the vector. Thus, the interval vector is made up of entries from the set $J = \{0, d/2, d\}$ where each element of J can appear at most once in the vector. Consequently, the vector has at most 3 entries, which implies that $c < 8$.

In fact, we can rule out some other chromatic cardinalities. The only cyclically generated scales in $c = 5$ and $c = 7$ universes are (1) the full chromatic scales ($g < c \Rightarrow (g,c) = 1$), and (2) the single element scales ($g = c$). However, none of these scales are DP: the former cases have duplicate d entries in their interval vectors, and the latter have duplicate zeros.

Turning now to $c = 6$, in order for a scale S in this universe to be DP, its interval vector must consist of all three elements of J. For d/2 to appear, S must contain a tritone. The only degenerate WF scales in $c = 6$ with tritones are of the forms $\{0, 3\}_6$ and $\{0, 1, 2, 3, 4, 5\}_6$. The interval vectors of these scales are [0 0 1] and [6 6 3], respectively, and thus neither possibility is DP.

We have shown that examples of F-set 6, if they exist, must be scales with $c < 5$. Let us now investigate such scales. Note that any S with a one element interval vector is trivially DP. Therefore, any degenerate WF scale with $c \leq 3$ is necessarily an example of F-set 6. There are five such scalar types summarized in the first five rows of Example 11.

The interval vectors of scales in $c = 4$ have two entries, the second of which represents the tritone. Therefore, any degenerate WF scale in $c = 4$ with a tritone is DP and belongs to F-set 6. This yields the last two scale types on Example 11. Note that $g = 4$, the only other possible cyclic generator, produces the

Example 11. A classification of all F-set 6 scales

c	d	g	Interval vector	Example
1	1	1	[0]	$\{0\}_1$
2	1	2	[0]	$\{0\}_2$
2	2	1	[1]*	$\{0,1\}_2$
3	1	3	[0]	$\{0\}_3$
3	3	1 or 2	[3]	$\{0,1,2\}_3$
4	2	2	[0 1]	$\{0,2\}_4$
4	4	1 or 3	[4 2]	$\{0,1,2,3\}_4$

(* = tritone)

set-class $\{0\}_4$, which has interval vector [0 0], and therefore scales of this type are not DP. It should also be noted that the DT scale in the c = 4 universe requires d = 3, so neither of the two c = 4 scales in Example 11 is DT.

We have shown that the seven scalar types listed in Example 11 exhaust all possibilities for F-set 6. While the examples are somewhat trivial, they are significant for purposes of completeness of the characterization of feature sets. Note also that the alternative characterization of DP given in Corollary 1.8 does not apply to scales that are this small. For example, the stipulation that d 5 c/2 or c/2 1 1 for even c leaves out the last scalar type on Example 11: c 5 d 5 4.

F-sets of infinite cardinality. Since it is possible to devise an algorithm to enumerate all pcsets in all chromatic universes, we might merely adjoin an appropriate sorting function to such an algorithm in order to enumerate the pcsets corresponding to any or all F-sets. Unfortunately, the design of such a procedure would confer little or no additional insight—although observing the sequence of its sorted output could be quite interesting, as suggested by Example 9. For the eleven F-sets with infinite memberships, however, we have developed algorithms that do yield insights as to the proclivities of the defined families—algorithms based primarily upon implicative relationships among features.

The basic methodology underlying the development of the algorithms was to begin with a "filtering" algorithm—essentially a sieve designed to capture desired properties and to filter out undesirable ones—based on the explicitly defined features of a given F-set and on the implicative relations presented above, and then to refine the filter, moving toward a more efficient "theorem-based" algorithm. Implicative relations were invoked at an early stage in the construction of these algorithms not only for reasons of efficiency, but in many cases out of necessity as well, since four of the properties, namely WF, ME, DE, and non-rounded MP (step sizes are nonconsecutive integers), cannot be characterized prescriptively in terms of restrictions on G-set parameters. This means that we are able neither to specify nor to directly eliminate from consideration the properties WF, ME, DE, or generic (non-rounded) MP. The ramifications of this non-commensurability quickly become apparent, as we shall soon see, when one attempts to differentiate between scales belonging to F-sets 7 or 10, or to F-sets 8 or 13—feature sets differentiated only by the presence or absence of generic MP, WF, and DE. Fortunately, since the five commensurably defined properties are situated, for the most part, near the top of the implicative network shown in Example 2, the remaining four non-commensurable features generally can be accounted for (positively or negatively) on the basis of implicative relationships.

As a general model of the filtering-type algorithm, the role played by implicative relationships among features, and the types of refinements involved in moving from a filtering-type algorithm toward a more theorem-based algorithm, consider the two algorithms for F-set 4 given in Example 12. F-set 4 includes the generic triad or seventh chord as embedded in the diatonic scale and also embraces all and only those sets with Agmon's (1991) efficient linear transformation property. Example 12a presents the final version of the non-terminating algorithm for this family of pcsets.[7] Given c=7, this particular algorithm clearly does generate the triad and seventh chord. It is by no means obvious, however, that it produces either all or only those pcsets that map onto F-set 4. That it does so becomes apparent if we examine the initial filtering form of the algorithm, given here as Example 12b, and then trace the refinements that led to the final version.

As shown in Example 3a, F-set 4 has all of the features except DT and BZ. Based on our network of implicative relationships, then, it should be sufficient to define F-set 4 as having MP, ME, and DP, but not DT or BZ. Or, since MP and ME together imply rounded MP, we can simply say that F-set 4 has rounded MP and DP, but not DT or BZ. Therefore, the algorithm in Example 12b is constructed such that for each value of c, values of d and g conducive to the presence of DP and rounded MP

[7]This algorithm is essentially contained in Agmon's (1991, 29–30) theorem pertaining to efficient sets of linear transformations.

Example 12. Algorithm for F-set 4

a. Final version

1. Set c = 3 (minimum value).
2. Set g = 2.
   Generate {0g, 1g, 2g, ... , c − 3} [d = (c − 1)/2 in this case].
   Generate {0g, 1g, 2g, ... , c − 1} [d = (c + 1)/2 in this case].
3. Set c = c + 2. Go back to step 2.

b. Earlier version

1. Set c = 3 (minimum value).
2. Choose d such that:
   For even values of c, d = c/2 or (c/2) + 1.
   For odd values of c, d = (c − 1)/2 or ((c − 1)/2) + 1.
3. For each pair (c, d),
   Find c′ such that cc′ ≡ −1 mod d.
   If some such value of c′ exists, find d′ such that d′ = (cc′ + 1)/d.
   Set g = d′.
   If c′ does not exist, discard the pair (c, d).
4. Discard all (c, d, g) where c ≡ 0 mod 4 and c = 2(d − 1).
5. Discard all (c, d, g) where c = k(k + 1) and d = g = 2k + 1 for some integer k (k > 2).
6. Generate {0g, 1g, 2g, ... , (d − 1)g} (all products taken mod c).
7. Set c = c + 1. Go back to step 2.

are selected—in steps 2 and 3, respectively—then any combinations of parameters that also result in DT or BZ are filtered out in steps 4 and 5.

While this algorithm is correct, in that it produces all pcsets in F-set 4 and no others, it is neither particularly efficient nor informative in terms of the sort of output we might expect. An important step toward both goals involves the exclusion of all even values of c, since these values always yield combinations of parameters that are filtered out in steps 3 or 4. Clearly step 4 excludes some pcsets with c even—those where c ≡ 0 mod 4 and d = c/2 + 1. Somewhat less obvious are the cases where c is even and d = c/2 or where c ≡ 2 mod 4 and d = c/2 + 1. Both are eliminated in step 3 since there does not exist a value of c′ that will fulfill the given conditions if c and d are not coprime. By restricting c to odd values only, then, we can disregard the "c is even" case in step 2, and—since an odd number cannot be congruent to 0 mod 4, nor can it be factored as k(k + 1)—we can eliminate steps 4 and 5 outright. This improved form of the algorithm can be further refined if we note that when c is odd step 3 returns g = 2 for both possible values of d. (We omit the proof here, but this can quite readily be shown to be the case.) Thus we can simply specify "set g = 2" and omit step 3 altogether. Our final revision to the algorithm involves a reworking of step 6. By rewriting d in terms of c—using both possible values of d given in step 2—and then simplifying, we find that the last pc generated is either c − 1 or c − 3, depending upon the value of d chosen. This information is incorporated into step 2 of the reformatted version of the algorithm given in Example 12a.

Exhaustive algorithms for F-sets 2, 3, 5, 7, 8, 10, and 13 can be constructed following similar procedures. Here we present only the final versions of these algorithms, accompanied by brief commentaries.

The non-terminating algorithm for F-set 2, which includes all diatonic scales except for the usual one, is given in Example 13. F-set 2 scales have seven of the eight features, lacking only BZ. On the basis of the implicative relationships, then, it is sufficient to define F-set 2 scales as having DT but not BZ. The algorithm is therefore constructed to generate all DT scales and then to filter out any scales that also have BZ. Steps 1 and 3 select all values of c such that c ≡ 0 mod 4, while step 2 identifies the values of d and g that will produce a DT-set for each c chosen. Then, since we have already shown that the usual diatonic is the only scale that's both DT and BZ, we need merely eliminate the case "c = 12" (in step 3) for the algorithm to produce all and only those pcsets belonging to F-set 2.

The algorithm for F-set 3, given in Example 14, is similarly straightforward. F-set 3 includes all non-diatonic BZ scales and can be characterized most simply as having BZ but not DT or

DP. Since BZ implies all other features except for DT and DP, the strategy underlying the construction of this algorithm was to generate all BZ scales and then to filter out any scales with DT and/or DP. Since we have already demonstrated that a scale cannot be both BZ and DP without also being DT, we need only filter out scales that are BZ and DT. Again, we have already shown that the usual diatonic is the only such scale, and it is excluded here by setting $k = 4$ as the minimum case in step 1.

F-set 5 presents a more complicated situation, in that its members possess only five of the eight properties, lacking in particular DT, DP, and BZ—three of the four commensurably defined properties located at the top of the implicative network. F-set 5 scales do have rounded MP (step sizes are consecutive integers), the fourth of these commensurable features; however, our definition of rounded MP in terms of G-set parameters does not prescribe values of c and d, but instead searches for viable values of g for each pair (c,d) under consideration. As a result, rather than simply prescribing desirable values of c, d, and g and then filtering out DT- and/or BZ-sets, as in the algorithms described above, here we must begin by considering all values of c, then for each c we must run all possible values of d through a series of filtering steps (we know which values of d we do not want—we do not want values that will result in DT scales, for example), then for each surviving pair (c,d) we can select a value of g that will guarantee a rounded MP-set. More specifically, steps 1 and 4 of the algorithm (given in Example 15) insure that all integer values of c ($c \geq 5$) will be considered. In step 2, appropriate values of d are selected by running all possible values of d through a series of filtering steps: the exclusion of $d = 1$ and of all values of d not co-prime with c eliminates values that produce (c,d) pairs that prove to be non-viable in step 3 (c' values do not exist for these pairs); values of d resulting in DP scales are filtered out in the steps "For odd values of c, $d \neq (c - 1)/2$ or $((c - 1)/2) + 1$" and "For even values of c, $d \neq (c/2) + 1$" (here the case "$d \neq c/2$" has already been excluded by the restriction $(c,d) = 1$); values of d producing DT scales are also excluded by the requirement that "For even values of c, $d \neq (c/2) + 1$"; and values of d resulting in BZ scales are filtered out by the restriction "For even values

Example 13. Algorithm for F-set 2

1. Let $c = 8$ (minimum value).
2. Set $d = g = (c/2) + 1$.
   Generate $\{0g, 1g, 2g, \ldots, (d - 1)g\}$ (all products taken mod c).
3. If $c = 8$, set $c = 16$; else set $c = c + 4$. Go back to step 2.

Example 14. Algorithm for F-set 3

1. Set $k = 4$ (minimum value).
2. Set $c = k(k + 1)$.
3. Set $d = g = 2k + 1$.
   Generate $\{0g, 1g, 2g, \ldots, (d - 1)g\}$ (all products taken mod c).
4. Set $k = k + 1$. Go back to step 2.

Example 15. Algorithm for F-set 5

1. Set $c = 5$ (minimum value).
2. Find all values of d ($d \leq c$) such that:
   $d > 1$.
   $(c,d) = 1$.
   For odd values of c,
      $d \neq (c - 1)/2$ or $((c - 1)/2) + 1$.
   For even values of c,
      $d \neq (c/2) + 1$, and
      if $c = k(k + 1)$ for some integer k ($k > 2$), then $d \neq 2k + 1$.
3. For each value of d,
   Find c' such that $cc' \equiv -1 \bmod d$ ($0 < c' < d$), then set $g = (cc' + 1)/d$.
   Generate $\{0g, 1g, 2g, \ldots, (d - 1)g\}$(all products taken mod c).
4. Set $c = c + 1$. Go back to step 2.

of c, if c = k(k + 1) for some integer k (k > 2), then d ≠ 2k + 1." For each remaining (c,d) pair, step 3 then selects a value of g resulting in a rounded MP scale.

As noted earlier, the lack of commensurable definitions for four of the eight features under consideration also affects the construction of algorithms for F-sets 7, 8, 10, and 13. Non-terminating algorithms capable of generating all pcsets belonging to F-sets 7 or 10 or to F-sets 8 or 13 can readily be produced following the same general methodology invoked above. Mapping each of these pcsets onto the single appropriate F-set would appear to be far more complicated, however, since F-sets 7 and 10, as well as F-sets 8 and 13, are differentiated only by the presence or absence of generic MP, WF, and DE—properties not defined *prescriptively* in terms of restrictions on G-set parameters. Fortunately, WF can be characterized in terms of the continued fraction approximations of the rational number "g/c," as described in Carey and Clampitt (1989), providing us with a mechanism for sorting pcsets within each pair of F-sets.

Consider the case of F-sets 7 and 10, for example. F-set 10, which includes all deep scales with the generic step appearing in three sizes, is characterized by G and DP only, while F-set 7, which includes all deep scales with all generic intervals appearing in two non-consecutive sizes, has G, DP, MP, WF, and DE. The algorithm for these two F-sets, given in Example 16, is constructed so as to first generate all pcsets belonging to the union of the F-sets, and then to sort these pcsets into the appropriate F-sets. For each value of c, step 2 produces those values of d required for deep scales, then for each resulting (c,d) pair, step 3 selects all values of g guaranteeing DP (in step 3.i) and eliminating rounded MP (3.ii) and DT (3.iii). Steps 4 and 5 then sort the G-set triples between the two F-sets. If g ≡ ±1 mod c, the scale represented by the triple (c,d,g) belongs to F-set 7. For all other values of g, "g/c" must be converted into a continued fraction and its convergents and semi-convergent must be found. If the value d appears in the denominator of any of these fractions, the scale represented by the triple (c,d,g) is WF and therefore belongs to F-set 7; otherwise, the triple is assigned to F-set 10. The algorithm for F-sets 8 and 13, given in Example 17, is structured in much the same way. We leave the examination of its specifics to the reader.

Example 16. Algorithm for F-sets 7 and 10

1. Set c = 7 (minimum value).
2. Find all values of d such that:
   for odd values of c, d = (c − 1)/2 or ((c − 1)/2) + 1.
   for even values of c, d = c/2 or (c/2) + 1.
3. For each pair (c, d), find all values of g such that:
   (i) (c, g) = 1.
   (ii) if there exists some value c′ (0 < c′ < d) such that cc′ ≡ −1 mod d, then g ≢ ±((cc′ + 1)/d) mod c, and
   (iii) if c ≡ 0 mod 4 and d=(c/2) + 1, then g ≢ ± ((c/2) + 1) mod c.
4. All scales represented by triples (c, d, g) with g ≡ ± 1 mod c belong to **F-set 7.**
5. For each remaining triple (c, d, g):
   Convert the rational number "g/c" into a continued fraction, then find all convergents and semiconvergents.
   If the value assigned to d appears in the denominator of any of these fractions, the scale represented by the triple (c, d, g) belongs to **F-set 7.**
   If the value assigned to d does not appear in the denominator of any of these fractions, the scale represented by the triple (c, d, g) belongs to **F-set 10.**
6. Set c = c + 1. Go back to step 2.

While a more intuitive approach was used to construct the algorithms for F-sets 9, 11, and 12, these algorithms also yield insights as to the idiosyncrasies of the defined families of pcsets, and point toward interesting relationships among the various properties discussed in this paper. These families include those pcsets most often discussed under the rubric of "interval cycles"—namely transpositional cycles and

Example 17. Algorithm for F-sets 8 and 13

1. Set c = 6 (minimum value).
2. Find all values of d such that:
   (i) $1 < d < (c - 1)$, and
   (ii) for even values of c, $d \neq c/2$ or $(c/2) + 1$, or
   for odd values of c, $d \neq (c - 1)/2$ or $((c - 1)/2) + 1$.
3. For each pair (c, d), find all values of g such that:
   (i) $(c, g) = 1$, and
   (ii) if there exists some value c' $(0 < c' < d)$ such that $cc' \equiv -1$ mod d, then $g \not\equiv \pm ((cc' + 1)/d)$ mod c.
4. For all triples (c, d, g) with $g \equiv \pm 1$ mod c,
   if $d = c - 2$, discard the triple;
   for all remaining values of d, the scale represented by the triple belongs to **F-set 8.**
5. For each remaining triple (c, d, g):
   Convert the rational number "g/c" into a continued fraction, then find all convergents and semiconvergents.
   If the value assigned to d appears in the denominator of any of these fractions, the scale represented by the triple (c, d, g) belongs to **F-set 8.**
   If the value assigned to d does not appear in the denominator of any of these fractions, the scale represented by the triple (c, d, g) belongs to **F-set 13.**
6. Set c = c + 1. Go back to step 2.

Example 18. Algorithm for F-set 9

1. Set c = 2 (minimum value).
2. Find all values of d such that $d \mid c$ $(1 < d \leq c)$.
3. For each value of d:
   Find the smallest value of g such that $dg \equiv 0$ mod c.
   Generate {0g, 1g, 2g, ..., (d − 1)g} (all products taken mod c).
4. Set c = c + 1. Go back to step 2.

cycle-combinations. That this is in fact the case can be intuited as follows.

Consider first the case of a pcset that has ME but not MP. As defined in Example 1, a maximally even set is one in which each generic interval comes in either one size or two consecutive integer sizes, and a Myhill property set is one in which each generic interval comes in two specific sizes. Thus, if a pcset is ME but not MP, at least one generic interval must come in one size only, implying an equal partitioning of the octave at some level—the level of the step, third, or etc. This leaves two possible cases to consider: either all generic intervals come in one size only, or at least one generic interval comes in one size only and at least one comes in two consecutive integer sizes.

In the first case, if all generic intervals come in one size only, the octave is partitioned at the level of the step. This, of course, produces the basic transpositional cycles with g = c/d, which correspond to F-set 9. The algorithm is given in Example 18.

In the second case, which corresponds to F-set 11, if at least one generic interval appears in one size only and at least one appears in two consecutive sizes, then the partitioning of the octave must occur at a level higher than that of the step since differentiated step sizes are necessary if any generic interval is to appear in more than one size. Furthermore, each partition must include an identical maximally even set with two step sizes—that is, a rounded MP-set—otherwise some generic interval will appear in three sizes. The characterization of these pcsets as being composed of smaller rounded MP sets disposed at equal intervals is consistent with our earlier discussion of an expanded understanding of "generated sets" and is reflected in the structure of the algorithm for F-set 11, given in Example 19.

F-set 12 can similarly be shown to include only those pcsets that are symmetrical under non-trivial transposition and under inversion, but that are not ME. If a set is DE but not ME, then either its generic intervals all come in two non-consecutive integer sizes or they include at least one interval appearing in

Example 19. Algorithm for F-set 11

Let a, b = step sizes; #a, #b = multiplicities of a, b (#a $\leq$ #b) within each segment; c′ = chromatic size of segment; d′ = diatonic size of segment; n = number of segments.

1. Set c = 6 (minimum value).
2. Find all values of d ($1 < d < c$) such that (c, d) $\neq$ 1.
3. For each value of d, find the common factor(s) n ($n > 1$) of c, d.
4. For each n, find c′ and d′ such that c′ = c/n and d′ = d/n.
5. For each d′, set #b = d′ − 1, and set #a = 1.
6. For each #b, find all solutions of a+(#b · b) = c′ where a and b are consecutive integers.
7. For each solution (a,b) found in Step 6, choose an ordering of step sizes a and b with multiplicities 1 and #b, respectively.
8. To produce the whole scale, replicate this segment n times at transposition levels 0, c′, 2c′, 3c′, ... , (n − 1)c′.
9. Set c = c + 1. Go back to step 2.

Example 20. Algorithm for F-set 12

Make the following changes to the F-set 11 algorithm:

1. Set c = 8 (minimum value).
6. Specify "non-consecutive" rather than "consecutive" integers.

one size only and at least one appearing in two non-consecutive sizes. The restriction "not MP" rules out the former possibility, leaving but one case: at least one interval must come in one size only and at least one must come in two non-consecutive sizes. As noted in conjunction with F-set 11, this implies an equal partitioning of the octave at a level higher than that of the step, with each partition in this case containing an identical non-rounded MP-set. Thus, the constituent pcsets of F-set 12 can also be characterized as "generated sets" in the expanded sense described above. The extent of the similarity between F-sets 11 and 12 is evident in the minor alterations needed to change the F-set 11 algorithm into that of F-set 12, as noted in Example 20.

F-sets 11 and 12 are differentiated, in fact, only by the actual step sizes involved (consecutive integer sizes in F-set 11, non-consecutive in F-set 12); their underlying structures, viewed in terms of the patterns of distribution of two unspecified step sizes, are identical. Given any pcset from F-set 11, for example, we can produce a pcset (in fact infinitely many pcsets in arbitrarily large universes) belonging to F-set 12 by increasing or decreasing one (or both) of the step-interval sizes so that they are not consecutive integers. This relationship, which we call "inflation" or "deflation," also obtains between F-sets 4 and 7 and between F-sets 5 and 8.

The inflation/deflation relationship is particularly interesting as it applies to the categorization of embedded scales. The "French Sixth," for example, is a member of F-set 12 when embedded in the usual twelve-note chromatic universe ($\{0, 2, 6, 8\}_{12}$, with $<1> = \{2, 4\}$) but belongs to F-set 11 when viewed in terms of a mod 6 universe ($\{0, 1, 3, 4\}_6$, with $<1> = \{1, 2\}$).

5. FURTHER REMARKS ON FEATURES

Complementation. In our characterization of the various features and their instantiations, we have yet to address the matter of complementation or, more specifically, to note which features hold for complements of sets. It might seem that, if a given set is generated, its complement is generated as well, by continued iteration of the same generating interval. For example, the usual diatonic and its pentatonic complement are both generated by interval 7 (or 5). However, this is the case only when the generating interval runs through all the pcs, which is to say when its size is co-prime to that of the universe. Four features always hold under complementation, namely MP, WF, ME, and DP. Of the remaining three features—DE, DT, and BZ—DE does not necessarily (but may) hold for complements, while DT and BZ never do. (That the latter assertion is true can readily be seen by examining the relationship between c and d in the definitions of these properties as given in Example 1.) Finally, it is curious that, while complementation preserves ME, it does not necessarily preserve DE. For example, the complement of the DE set {0, 1, 6, 7} is also DE, while the complement of {0, 2, 6, 8}, the "French Sixth," is not. Note that neither of these sets is ME.

Cohn's property. We now return briefly to Cohn's property, that remarkable feature shared by the usual diatonic, the consonant triads, and infinitely many sets in spaces of other than 12 pcs—the feature that enables formation of cycles of pcsets of the same set-class by means of a single minimal motion of one pc. In 12-pc space, these *maximally smooth cycles,* as Cohn calls them, are exemplified, non-trivially, by the circle-of-fifths arrangement of the twelve diatonic scales, and by the four cycles of six consonant triads each (for example, C major, C minor, A♭ major, A♭ minor, E major, E minor, then back to C major), and by cycles of the complements of those sets. In 7-space, they are exemplified by a circle-of-3ds progression of the triads. How does Cohn's property relate to our classification? We conjecture that all pcsets in F-sets 1–5 have Cohn's property. These F-sets comprise precisely all the sets with rounded MP. We conjecture further that these are the only pcsets within our classification that have Cohn's property. Merging these two statements, we suspect that, in the presence of G, rounded MP and Cohn's property are mutually implicative. (As far as we know, these conjectures have not been proved. If and when they are, it seems likely that the method of proof will owe a debt to David Lewin's (1996) work describing the necessary structure of sets with Cohn's property.) In summary, within the confines of our taxonomy—that is, within the world of generated sets as we recognize them here—Cohn sets are nicely circumscribed. There are, however, infinitely many sets with maximally smooth cycles that are not generated sets and therefore lie outside our classification.

Distributionally even sets. Before concluding, we offer further remarks on the category we have called "distributionally even," beginning with an account of some of its more visible antecedents. These extend back at least to Messiaen's 1944 compositional treatise with its discussion of his "modes of limited transposition" and range forward to Howard Hanson's 1960 study devoted in small part to pcsets based on dual interval cycles. More recently, a line may be traced, more or less directly from the 1977–78 work of Starr and Morris, through Morris's 1987 treatise, through Cohn's 1991 paper, to our DE category. Along the way, one finds the 1982 paper of Lewin, whose generalized Riemann Systems are necessarily conceivable as dual interval cycles, possibly incomplete, and the 1995 characterization scheme of Anatol Vieru, based on "knots" of incomplete interval cycles. Two recent contributions relating to the DE property are Stephen Soderberg's 1995 study of Z-sets and Jay Rahn's 1996 paper in which DE-sets are studied in the rhythmic domain. In resonance with Richmond Browne's 1981 discussion of the unique interval contexts for each pc in the usual diatonic, Rahn characterizes pcsets such as the octatonic scale as sets where one may "look out" from more than one pc (but not all) and see the same "panorama." This metaphor perfectly describes the subcategory of our DE scales in which at least one generic interval comes in one size and at least one comes in two sizes.

So there is an ample body of literature addressing Messiaen's modes, transpositionally redundant sets in general, and the broad notion of the dual or multiple interval cycle. Our contribution along these lines is to define the category "distributionally even" in terms of generic interval sizes, and to open the category to cases where specific interval sizes are not rational parts of the octave. In so doing, we define a category that subsumes all the others in our classification except the deep scales and the generated scales with no other features. Like the category maximally even, it catches the pentatonic, whole-tone, diatonic, and octatonic sets; in addition, it includes all conjunctions of two complete interval cycles based on the same generator and many of more than two such cycles, thus embracing most of Messiaen's modes and, more significantly, tunings of all the above that hold constant the distribution of relative interval sizes, for example, the Pythagorean diatonic.

The great generality afforded by the category DE leads us to suspect that further attempts to define categories in terms of generic and specific intervals (perhaps involving cases with more than two specific sizes for some generic sizes) may yield interesting results, but we leave that for the future.

Conclusion. Our principal objective has been to unify the contributions of many theorists who, motivated by a sense of wonderment at the diatonic scale, its durability, adaptability, and ubiquity, have taken account of its structure, and have gone on to pose questions extending to various musical dimensions, diverse musics, and the nature of musical systems, both extant and imaginary. We hope that the present effort will encourage and facilitate their continuing quest.

ABSTRACT

Recent studies in the theory of scales by Agmon, Balzano, Carey and Clampitt, Clough and Douthett, Clough and Myerson, and Gamer have in common the central role of the interval cycle. Based on scale features defined in these studies, and an additional feature called *distributional evenness* defined here, a taxonomy is proposed for pitch-class sets (pcsets) the correspond to interval cycles or to certain conjunctions thereof. Pairwise implicative relationships among the features are explored. Of twenty sets of features that are consistent with these relationships, thirteen are found to be instantiated by actual pcsets and seven others are shown to be incapable of instantiation. Most instantiated feature-sets correspond to infinite classes of pcsets which are shown to be enumerable; one such feature-set is found to be uniquely realized (up to transposition) in the usual diatonic pcset.

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