SOME NEW LINEAR TEMPERAMENTS

...John Chalmers, Oct. 1974.

From the 16th. to the 19th. century, organs and other keyboard instruments were tuned to some version of meantone temperament. This tuning system was gradually replaced by various irregular varieties, as documented in Barbour's "Tuning and Temperament", and finally yielded to the familiar 12-tone equal temperament. Recently, however, there has been a resurgence of interest in performing music of this period in its authentic intonation. For this reason, one might profitably reexamine unequal or linear temperaments, not only for the performance of this music, but more for the novel properties of these systems.

Tunings of the meantone type are characterised by a unique size of the fifth whose cycles, of course, do not common out evenly with the octave. Depending upon the size chosen for the fifth, certain other intervals may have their just values. In the case of meantone itself, the major third is exactly 5/4, though the minor third and fifth are quite far from their true measures. The Third-Comma system of Salinas has true minor thirds and major sixths, while the 1/5-Comma tuning has pure major sevenths. These and other historically proposed temperaments are dealt with at length by Barbour. In addition to offering more harmonious triads than 12-tone equal temperament, other intervals such as 7/4, 7/5 and 7/6 are also approximated in some chords. The greatest disadvantage of unequal temperaments is the restriction on free modulation, or extended chromatic writing, although this point overstressed. During the period of their employment, these systems were prized by many composers precisely because they did distinguish between keys, giving to modulation some of the effect of a change of mode. In any case, there are equal temperaments closely corresponding to each of the meantone varieties.

In contrast to the meantone type of temperament where the fifths are less than those of 12-tone equal, there are also positive systems with fifths larger than 700 cents. The prototype of these systems is the Pythagorean with its just fifths. Positive systems can also be used to approximate other intervals, although the chain is sometimes rather long. For example, meantone or negative systems form their major third by going four fifths up from the tonic and subtracting two octaves. Positive systems form their major third (C-Fb) by going eight fifths down and five octaves up. Similarly, the harmonic seventh (7/4) is found in negative systems by ten ascending fifths; the positive require fourteen descending steps. Doubly positive systems also exist which are similar to the 22-tone equal temperament. In these systems, rather unfamiliar relationships are encountered.

One can appreciate that the historical linear temperaments were designed to express triadic relations in a limited number of tones. They, furthermore, had the defect that the errors tended to be concentrated in the fifth or third, and the intonation of the higher prime intervals was ignored. An attempt has been made to correct these deficiencies by designing new linear temperaments. Clearly, we are no longer limited conceptually to tertian intervals, nor really to twelve keys per octave anymore. The development of electronic pitch control and microtonal keyboards makes the positive systems both attractive and feasible. New negative systems have been found which balance the tertian intervals more equitably and which also rectify the more important higher prime relations. These likewise are conceived in a microtonal context, though many would be excellent for traditional harpsichords and organs.

As a first step in this investigation, a FØRTRAN program was written to calculate the number of fifths of different sizes which generated certain just intervals. The size of the fifth was allowed to range from 690 to 720 cents, and the length of the cycle could go as high as fifty-three steps up or down. It was found that cycles of reasonable length were defined by fifths of three sizes-- a meantone-like negative fifth, a positive fifth near the just value, and with more searching, a fifth near to the 22-tone tempered value. These values, for the more important intervals, have been tabulated in Table 1. Also included in this table are the values for certain historical systems and some closely related equal temperaments.

Once the size of the fifth and the length of the cycle defining a given interval is known, it becomes a simple task to write functions describing the errors in these intervals in terms of the size of the fifth. In the 1/3-Comma system, for example, the errors of the major third and fifth are equal. If we call the fifth F (701.9550+) and the major third T (386.3137+), the corresponding errors in any negative system can be written F-X and T-4X+2Ø, where Ø is the octave(1200) and X the new fifth. Thus the 1/3-Comma system is defined by F-X=T-4X+2Ø. temperament. By negating one of the functions, the equal and opposite system is obtained. This system is the 1/5-Comma system where the 15/8 has its just value. In Table 1., these two temperaments are written "3=5" and "3op5". This principle is extended to other intervals and the corresponding generating and error functions are given in Table 2. Positive and doubly positive systems are derived in the same manner.

There are innumerable ways in which the error functions can be combined. Various means, arithmetic, harmonic, geometric, to name the simplest, may be used. The technique called the Method of Least Squares was the one used most in this investigation. This technique finds tunings in which the total $(\text{error})^2$ of a given set of intervals is minimised. In the cases examined here, the solutions are weighted in favor of intervals with the longest chains of fifths. In practice this means that the seventh is favored over the third, and the eleventh over the seventh, but since large errors make the most contribution to the squared terms, the errors, in fact, are somewhat equalised. Intuitively, then, this would seem to be a good method. Since this technique does not distinguish between sharp and flat intervals, a similar experiment was tried using the absolute values of the errors. A simple one-dimensional search program was written and run, but no new tunings emerged--only the fifths corresponding to the intervals with the longest chains.

The application of the Method of Least Squares to this type of problem requires some explanation. Let us express the total error for a set of intervals approximated by a linear temperament by the following equations, using, for example, the fifth and major third of a negative system.

$$\text{Total Error} = (F-X) + (T-4X+2\emptyset), \quad \text{where } \emptyset \text{ is the octave or 1200 cents.}$$

$$\text{Total } (\text{Error})^2 = (F-X)^2 + (T-4X+2\emptyset)^2$$

Now the expression for the total squared error must be differentiated with respect to the desired fifth, X.

$$\frac{d(\text{Total } (\text{Error})^2)}{dX} = -2(F-X) - 8(T-4X+2\emptyset)$$

The derivative is now set equal to zero and the equation solved for X.

$$-2(F-X) - 8(T-4X+2\emptyset) = 0$$

$$X = 696.8947 \text{ cents.}$$

For comparison, the meantone fifth is 696.5714 cents, admitedly very close. The major third in this tuning, LSQ3,5, is 387.5788 cents, compared to the just value of 386.3137. The squared error for both the fifth and the major third is 27.2072 for the new tuning and 28.91 for meantone.

Other functions may be obtained from Table 2 and the process applied to discover tunings of this type. A number of these have been calculated and listed in Table 3., in each of the categories, negative, positive and doubly positive. Space, unfortunately, does not permit a detailed error analysis of each entry. However, it is not difficult to do given a table of just intervals and the generating functions.

Although much of this article may seem to be an exercise in speculative music theory, the new tunings presented here do appear to have intriguing properties. The subtle tunings can be realised with the aid of electronic pitch standards, and I have no doubt that a tuning order can worked out for many of them. It might prove interesting to temper the temperament along the line of Kuhnle's or Barbour's regularly varied schemes, remembering that intervals of 7 and other high primes can be obtained from chains of altered fifths. Perhaps the major importance of this work lies in the support certain of the new tunings give to the 31 and 41 tone temperaments. The 19, 31, 43, and 50 tone systems have long been known as ideal forms of certain negative systems, much as the 53-tone system has been the idealised form of Pythagorean. Although Wilson has developed just constructions for 22 and 41, this is the first time that the corresponding linear temperaments have been described. Thus the major harmonic equal temperaments may be referred to both just and linear constructions, both of which imply different musical usages--the just being the static and the linear the dynamic aspect of the harmonic derivation.

Rahway, New Jersey
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TABLE 1.

LINEAR AND EQUAL TEMPERAMENTS

Negative Systems	Fifths in Cents
19-Tone Equal	694.7368
31-Tone Equal	696.7742
43-Tone Equal	696.6744
50-Tone Equal	696.0000
55-Tone Equal	698.1818
Meantone, $\frac{1}{4}$-Comma, 5/4 Just	696.5784
1/3-Comma, 3=5, 6/5 Just*	694.7862
2/7-Comma, 25/24 Just *	695.8103
1/5-Comma, 3op5, 15/8 Just*	697.6537
1/6-Comma, 45/32 Just*	698.3706
7/4 NEG, 7/4 Just	696.8826
11/8 NEG, 11/8 Just	697.2954
13/8 NEG, 13/8 Just	696.0352
3=7, 7/6 Just	696.3190
3op7, 21/16 Just	697.3437
5=7, 7/5 Just	697.0854
5op7, 35/32 Just	696.7957
11/9-16up, 11/9 Just	696.7130

Positive Systems	
41-Tone Equal	702.4390
53-Tone Equal	701.8868
Pythagorean, 3/2 Just	701.9550
Helmholtz, 5/4 Just	701.7108
3=5, 6/5 Just	701.7379
3op5, 15/8 Just	701.6759
7/4 POS, 7/4 Just	702.2267
11/8 POS, 11/8 Just	702.7046
13/8 POS, 13/8 Just	702.8320
3=7, 7/6 Just	702.2086
3op7, 21/16 Just	702.2476
5=7, 7/5 POS, 7/5 Just	702.9146
5op7, 35/32 Just	702.0391
11=7, 11/7 Just	701.0942
19/16 POS, 19/16 Just	700.8290

Doubly Positive Systems	
22-Tone Equal	709.0909
5/4-2POS, 5/4 Just	709.5904
7/4-2POS, 7/4 Just	715.5870
11up16, 11/8 Just	709.4573
11dn6, 11/8 Just	708.1137
13/8-2POS, 13/8 Just	710.8098
3=5, 6/5 Just	710.5448
3op5, 15/8 Just	708.8269
3=7, 7/6 Just	711.0430
3op7, 21/16 Just	729.2191
5=7, 7/5 Just	710.6807
5op7, 35/32 Just	707.8771
11=9, 11/9 Just	710.5291
9=7, 9/7 Just	708.7710

* See Barbour, "Tuning and Temperament", for sources.

TABLE 2.

FUNCTIONS

INTERVALS	GENERATING FUNCTIONS	ERROR FUNCTIONS
Negative Systems		
Fifth	X	F-X *
Major Third (5/4)	4X-2Ø	T-4X+2Ø
Harmonic Seventh (7/4)	10X-5Ø	S-10X+5Ø
Harmonic Fourth (11/8)	18X-10Ø	E-18X+10Ø
Harmonic Sixth (13/8)	15X-8Ø	Th-15X+8Ø
Positive Systems		
Fifth	X	F-X
Major Third (5/4)	5Ø-8X	T+8X-5Ø
Harmonic Seventh (7/4)	9Ø-14X	S+14X-9Ø
Harmonic Fourth (11/8)	11Ø-18X	E+18X-11Ø
Harmonic Sixth (13/8)	13Ø-21X	Th+21X-13Ø
Doubly Positive Systems		
Fifth	X	F-X
Major Third (5/4)	9X-5Ø	T-9X+Ø
Harmonic Seventh (7/4)	2Ø-2X	S+2X-2Ø
Harmonic Fourth (11/8)	4Ø-6X	E+6X-4Ø
Harmonic Fourth (11/8)	16X-9Ø	E-16X+9Ø
Harmonic Sixth (13/8)	13X-7Ø	Th-13X+7Ø
Neutral Third (11/9)	14X-8Ø	N-14X+8Ø
Supramajor Third (9/7)	4X-2Ø	St-4X+2Ø

* The symbols in these expressions stand for the following just intervals in cents:

F = 701.955001 T = 386.313714 S = 968.825906
E = 551.317942 Th= 840.527662 N = 347.407941
St=435.084095 Ø = 1200.000000.

TABLE 3.

LEAST SQUARES SOLUTIONS

Negative Systems	Fifths in Cents
LSQ 3,5	696.8947
LSQ 3,7	696.9328
LSQ 5,7	696.8406
LSQ 3,5,7	696.8843
LSQ 3,5,7,11	697.1864
LSQ 3,5,7,11,13	696.7975
LSQ 5,7,11	696.1755
Positive Systems	
LSQ 3,5	701.7145
LSQ 3,7	702.2253
LSQ 5,7	702.0997
LSQ 3,5,7	702.0992
LSQ 3,5,7,11	702.4345
LSQ 3,5,7,11,13	702.6053
Doubly Positive Systems	
LSQ 3,5	709.4973
LSQ 3,7	712.8606
LSQ 5,7	709.8726
LSQ 3,5,7	709.7805
LSQ 3,5,7,11 (6X)	709.2887 *
LSQ 3,5,7,11 (16X)	709.5386 *
LSQ 3,5,7,11,13 (6X)	710.1721 *
LSQ 3,5,7,11,13 (16X)	709.9590 *

* See Table 2. for the two alternate functions of 11/8. The notation, (6X) or (16X), refers to the length of the chain of fifths which defines this interval.